Sine-Gordon Quarks and Pion

- Frank D. (Tony) Smith, Jr.
- http://www.valdostamuseum.org/hamsmith/
- June 2006

Abstract: The charged pion, made up of Up plus antiDown or Down plus antiUp quarks, is described in terms of the sine-Gordon and massive Thirring models. When the quark and antiquark are seen as Kerr-Newman black holes each having constituent mass about 312 MeV, the pion is seen as resulting from their merger, whch produces a black hole with toroidal event horizon representing a sine-Gordon meson whose mass can be calculated. The charged pion mass calculation gives a charged pion mass of about 139 MeV, which is substantially consistent with experimental results.

The quark content of a charged pion is a quark - antiquark pair: either Up plus antiDown or Down plus antiUp. Experimentally, its mass is about 139.57 MeV.

> The quark is a Naked Singularity Kerr-Newman Black Hole, with electromagnetic charge e and spin angular momentum J and constituent mass M 312 MeV, such that e^2 + a^2 is greater than M^2 (where a = J / M).
>
> The antiquark is a also Naked Singularity Kerr-Newman Black Hole, with electromagnetic charge e and spin angular momentum J and constituent mass M 312 MeV, such that e^2 + a^2 is greater than M^2 (where a = J / M).

According to General Relativity, by Robert M. Wald (Chicago 1984) page 338 [Problems] ... 4. ...:

> "... Suppose two widely separated Kerr black holes with parameters (M1 , J1) and (M2 , J2) initially are at rest in an axisymmetric configuration, i.e., their rotation axes are aligned along the direction of their separation.
>
> Assume that these black holes fall together and coalesce into a single black hole.
>
> Since angular momentum cannot be radiated away in an axisymmetric spacetime, the final black hole will have momentum J = J1 + J2. ...".

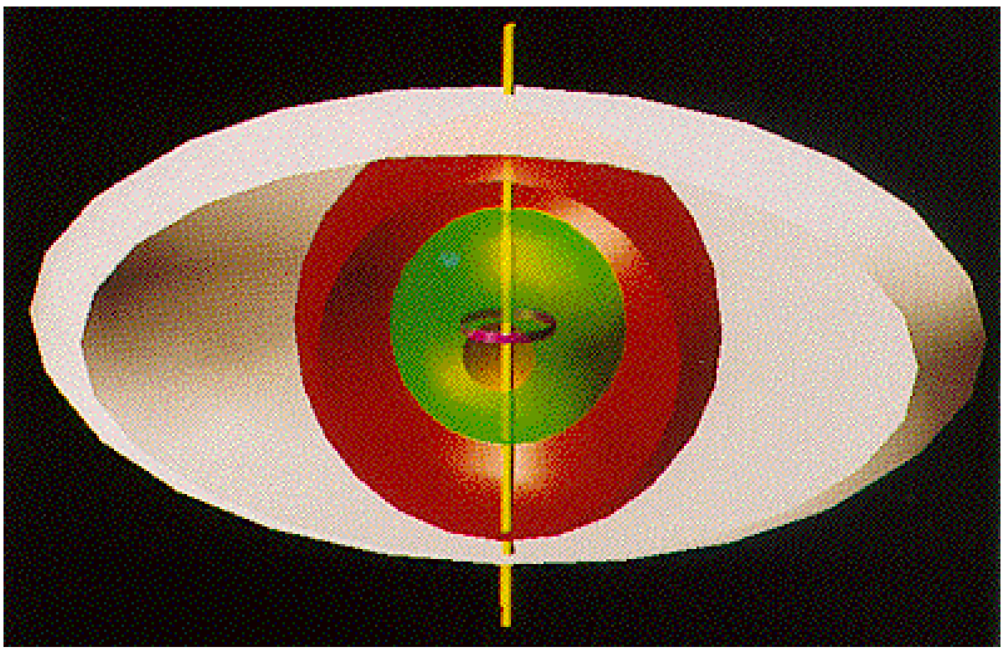
The neutral pion produced by the quark - antiquark pair would have zero angular momentum, thus reducing the value of e^2 + a^2 to e^2 .

For fermion electrons with spin 1/2, 1 / 2 = e / M (see for example Misner, Thorne, and Wheeler, Gravitation (Freeman 1972), page 883) so that M^2 = 4 e^2 is greater than e^2 for the electron. In other words, the angular momentum term a^2 is necessary to make e^2 + a^2 greater than M^2 so that the electron can be seen as a Kerr-Newman naked singularity.

Since the magnitude of electromagnetic charge of each quarks or antiquarks less than that of an electron, and since the mass of each quark or antiquark (as well as the pion mass) is greater than that of an electron, and since the quark - antiquark pair (as well as the pion) has angular momentum zero, the quark - antiquark pion has M^2 greater than e^2 + a^2 = e^2.

(Note that color charge, which is nonzero for the quark and the antiquark and is involved in the relation M^2 less than sum of spin-squared and charges-squared by which quarks and antiquarks can be see as Kerr-Newman naked singularities, is not relevant for the color-neutral pion.)

Therefore, the pion itself is a normal Kerr-Newman Black Hole with Outer Event Horizon = Ergosphere at r = 2M (the Inner Event Horizon is only the origin at r = 0) as shown in this image



from Black Holes - A Traveller's Guide, by Clifford Pickover (Wiley 1996) in which the Ergosphere is white, the Outer Event Horizon is red, the Inner Event Horizon is green, and the Ring Singularity is purple. In the case of the pion, the white and red surfaces coincide, and the green surface is only a point at the origin.

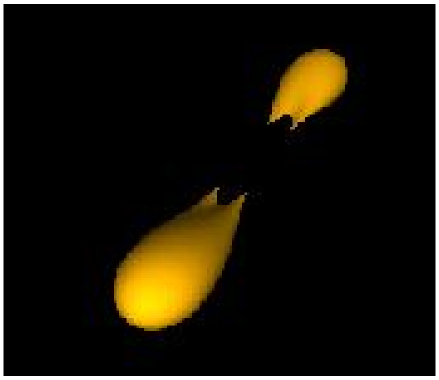
According to section 3.6 of Jeffrey Winicour's 2001 Living Review of the Development of Numerical Evolution Codes for General Relativity (see also a 2005 update):

> "... The black hole event horizon associated with ... slightly broken ... degeneracy [of the axisymmetric configuration]... reveals new features not seen in the degenerate case of the head-on collision ... If the degeneracy is slightly broken, the individual black holes form with spherical topology but as they approach, tidal distortion produces two sharp pincers on each black hole just

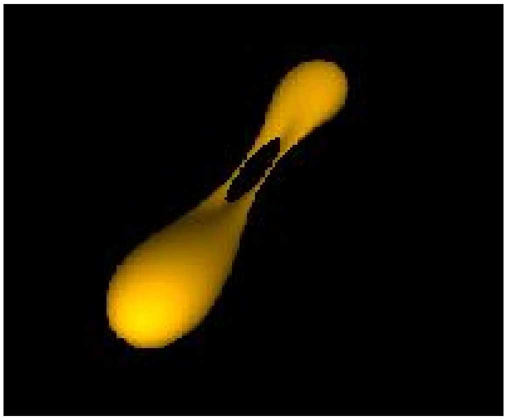
prior to merger.

... Tidal distortion of approaching black holes ...

... Formation of sharp pincers just prior to merger ..



... toroidal stage just after merger ...



At merger, the two pincers join to form a single ... toroidal black hole.

The inner hole of the torus subsequently [begins to] close... up (superluminally) ... [If the closing proceeds to completion, it]... produce[s] first a peanut shaped black hole and finally a spherical black hole. ...".

In the physical case of quark and antiquark forming a pion, the toroidal black hole remains a torus. The torus is an event horizon and therefore is not a 2-spacelike dimensional torus, but is a (1+1)-dimensional torus with a timelike dimension.

The effect is described in detail in Robert Wald's book General Relativity (Chicago 1984). It can be said to be due to extreme frame dragging, or to timelike translations becoming spacelike as though they had been Wick rotated in Complex SpaceTime.

As Hawking and Ellis say in The LargeScale Structure of Space-Time (Cambridge 1973):

> "... The surface r = r+ is ... the event horizon ... and is a null surface ...

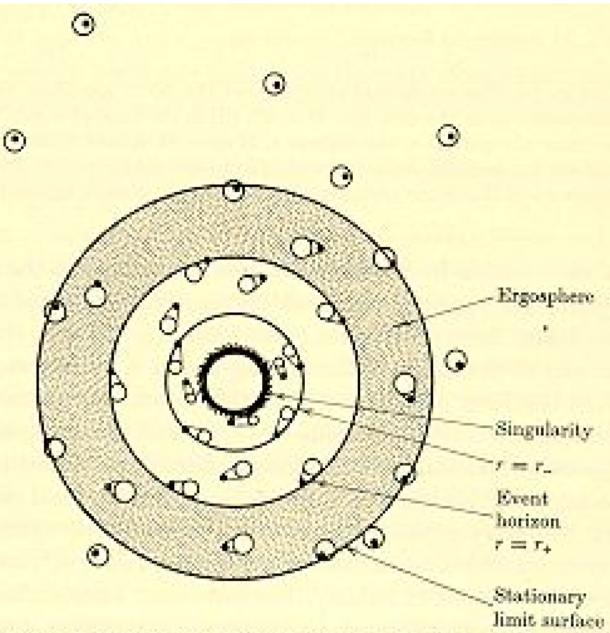


FIGURE 30. The equatorial plane of a Kerr solution with $m^2 > a^2$. The circles represent the position a short time later of flashes of light emitted by the points represented by heavy dots.

> ... On the surface r = r+ .... the wavefront corresponding to a point on this surface lies entirely within the surface. ...".

A (1+1)-dimensional torus with a timelike dimension can carry a Sine-Gordon Breather, and the soliton and antisoliton of a Sine-Gordon Breather correspond to the quark and antiquark that make up the pion.

Sine-Gordon Breathers are described by Sidney Coleman in his Erica lecture paper Classical Lumps and their Quantum Descendants (1975), reprinted in his book Aspects of Symmetry (Cambridge 1985), where Coleman writes the Lagrangian for the Sine-Gordon equation as (Coleman's eq. 4.3):

L = (1 / B^2) ((1/2) (df)^2 + A (cos(f) - 1)) and Coleman says:

"... We see that, in classical physics, B is an irrelevant parameter: if we can solve the sine-Gordon equation for any non-zero B, we can solve it for any other B. The only effect of changing B is the trivial one of changing the energy and momentum assigned to a given soluition of the equation. This is not true in quantum physics, becasue the relevant object for quantum physics is not L but [eq. 4.4]

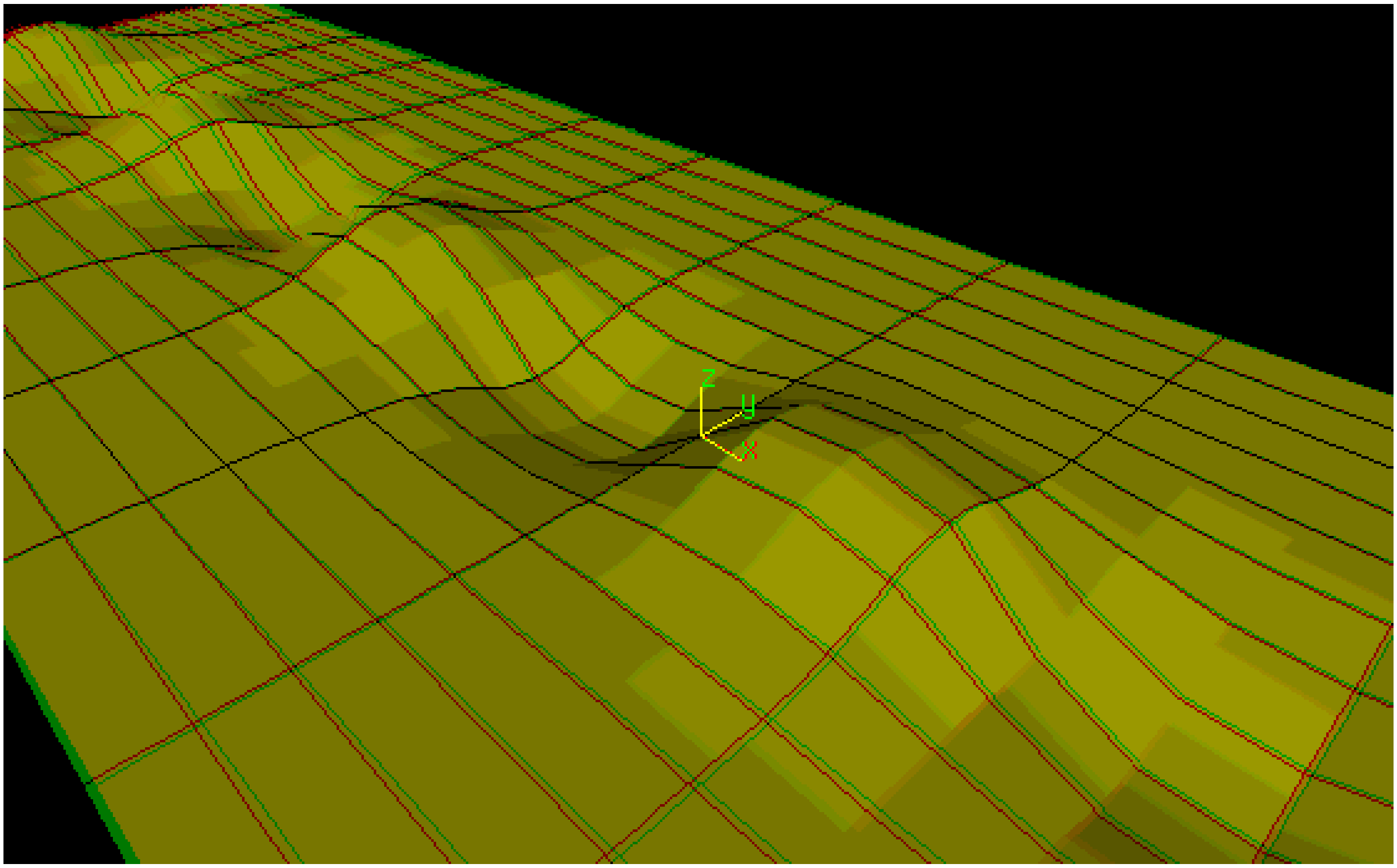
L / hbar = (1 / (B^2 hbar)) ((1/2) (df)^2 + A (cos(f) - 1))

An other way of saying the same thing is to say that in quantum physics we have one more dimensional constant of nature, Planck's constant, than in classical physics. ... the classical limit, vanishingf hbar, is exactly the same as the small-coupling limit, vanishing B ... from now on I will ... set hbar equal to one. ...

... the sine-Gordon equation ...[has]... an exact periodic solution ...[eq. 4.59]...

f(x, t) = (4 / B) arctan((n sin(w t) / cosh(n w x))

where [eq. 4.60] n = sqrt(A - w^2) / w and w ranges from 0 to A. This solution has a simple physical interpretation ... a soliton far to the left ...[and]... an antisoliton far to the right. As sin (w t) increases, the soliton and antisoliton mover farther apart from each other. When sin(w t) passes thrpough one, they turn around and begin to approach one another. As sin(w t) comes down to zero ... the soliton and antisoliton are on top of each other ... when sin(w t) becomes negative .. the soliton and antisoliton have passed each other. ...[



This stereo image of a Sine-Gordon Breather was generated by the program 3D-Filmstrip for Macintosh by Richard Palais. You can see the stereo with red-green or red-cyan 3D glasses. The program is on the WWW at http://rsp.math.brandeis.edu/3D-Filmstrip. The Sine-Gordon Breather is confined in space (y-axis) but periodic in time (x-axis), and therefore naturally lives on the (1+1)-dimensional torus with a timelike dimension of the Event Horizon of the pion. ...]

... Thus, Eq. (4.59) can be thought of as a soliton and an antisoliton oscillation about their common center-of-mass. For this reason, it is called 'the doublet [or Breather] solution'. ... the energy of the doublet ...[eq. 4.64]

$$E = 2 M \mathrm{sqrt}(1 - (w^2 / A))$$

where [eq. 4.65] $M = 8 \mathrm{sqrt}(A) / B^2$ is the soliton mass. Note that the mass of the doublet is always less than twice the soliton mass, as we would expect from a soltion-antisoliton pair. ... Dashen, Hasslacher, and Neveu ... Phys. Rev. D10, 4114; 4130; 4138 (1974). A pedagogical review of these methods has been written by R. Rajaraman (Phys. Reports 21, 227 (1975 ... Phys. Rev. D11, 3424 (1975) ...[Dashen, Hasslacher, and Neveu found that]... there is only a single series of bound states, labeled by the integer N ... The energies ... are ... [eq. 4.82]

$$E_N = 2 M \sin(B'^2 N / 16)$$

where $N = 0, 1, 2 \ldots < 8 \pi / B'^2$, [eq. 4.83]

$$B'^2 = B^2 / (1 - (B^2 / 8 \pi))$$

and M is the soliton mass. M is not given by Eq. (4.675), but is the soliton mass corrected by the DHN formula, or, equivalently, by the first-order weak coupling expansion. ... I have written the equation in this form .. to eliminate A, and thus avoid worries about renormalization conventions. Note that the DHN formula is identical to the Bohr-Sommerfeld formula, except that B is replaced by B'. ... Bohr and Sommerfeld['s] ... quantization formula says that if we have a one-parameter family of periodic motions, labeled by the period, T, then an energy eigenstate occurs whenever [eq. 4.66]

$$\int_0^T dt \; p \, \dot{q} = 2 \pi N,$$

where N is an integer. ... Eq.(4.66) is cruder than the WKB formula, but it is much more general; it is always the leading approximation for any dynamical system ... Dashen et al speculate that Eq. (4.82) is exact. ...

the sine-Gordon equation is equivalent ... to the massive Thirring model. This is surprising, because the massive Thirring model is a canonical field theory whose Hamiltonian is expressedin terms of fundamental Fermi fields only. Even more surprising, when $B^2 = 4 \pi$, that sine-Gordon equation is equivalent to a free massive Dirac theory, in one spatial dimension. ... Furthermore, we can identify the mass term in the Thirring model with the sine-Gordon interaction, [eq. 5.13]

$$M = - (A / B^2) N_m \cos(B f)$$

.. to do this consistently ... we must say [eq. 5.14]

$$B^2 / (4 \pi) = 1 / (1 + g / \pi)$$

....[where]... g is a free parameter, the coupling constant [for the Thirring model]... Note that if $B^2 = 4 \pi$, $g = 0$, and the sine-Gordon equation is the theory of a free massive Dirac field. ... It is a bit surprising to see a fermion appearing as a coherent state of a Bose field. Certainly this could not happen in three dimensions, where it would be forbidden by the spin-statistics theorem. However, there is no spin-statistics theorem in one dimension, for the excellent reason that there is no spin. ... the lowest fermion-antifermion bound state of the massive Thirring model is an obvious candidate for the fundamental meson of sine-Gordon theory. ... equation (4.82) predicts that all the doublet bound states disappear when B^2 exceeds 4π . This is precisely the point where the Thirring model interaction switches from attractive to repulsive. ... these two theories ... the massive Thirring model .. and ... the sine-Gordon equation ... define identical physics. ... I have computed the predictions of ...[various]... approximation methods for the ration of the soliton mass to the meson mass for three values of B^2 : 4π (where the qualitative picture of the soliton as a lump totally breaks down), 2π, and π . At 4π we know the exact answer ... I happen to know the exact answer for 2π, so I have included this in the table. ...

Method	B^2 = pi	B^2 = 2 pi	B^2 = 4 pi
Zeroth-order weak coupling expansion eq2.13b	2.55	1.27	0.64
Coherent-state variation	2.55	1.27	0.64
First-order weak coupling expansion	2.23	0.95	0.32
Bohr-Sommerfeld eq4.64	2.56	1.31	0.71
DHN formula eq4.82	2.25	1.00	0.50
Exact	?	1.00	0.50

...[eq. 2.13b] E = 8 sqrt(A) / B^2 ...[is the]... energy of the lump ... of sine-Gordon theory ... frequently called 'soliton...' in the literature ... [Zeroth-order is the classical case, or classical limit.] ...

... Coherent-state variation always gives the same result as the ... Zeroth-order weak coupling expansion ... .

The ... First-order weak-coupling expansion ... explicit formula ... is (8 / B^2) - (1 / pi). ...".

Note that,

using the VoDou Physics constituent mass of the Up and Down quarks and antiquarks, about 312.75 MeV, as the soliton and antisoliton masses,

and setting B^2 = pi

and using the DHN formula,

the mass of the charged pion is calculated to be (312.75 / 2.25) MeV = 139 MeV

which is in pretty good agreement with the experimental value of about 139.57 MeV.

Why is the value B^2 = pi (or, using Coleman's eq. (5.14), the Thirring coupling constant g = 3 pi) the special value that gives the pion mass ?

Because B^2 = pi is where the First-order weak coupling expansion substantially coincides with the (probably exact) DHN formula. In other words,

The physical quark - antiquark pion lives where the first-order weak coupling expansion is exact.

Near the end of his article, Coleman expressed "Some opinions":

> "... This has been a long series of physics lectures with no reference whatsoever to experiment. This is embarrassing.
>
> ... Is there any chance that the lump will be more than a theoretical toy in our field? I can think of two possiblities.
>
> One is that there will appear a theory of strong-interaction dynamics in which hadrons are thought of as lumps, or, ... as systems of quarks bound into lumps. ... I am pessimistic about the success of such a theory. ... However, I stand ready to be converted in a moment by a convincing computation.
>
> The other possibility is that a lump will appear in a realistic theory ... of weak and electromagnetic interactions ... the theory would have to imbed the U(1)xSU(2) group ... in a larger group without U(1) factors ... it would be a magnetic monopole. ...".

This description of the hadronic pion as a quark - antiquark system governed by the sine-Gordon - massive Thirring model should dispel Coleman's pessimism about his first stated possibility and relieve his embarrassment about lack of contact with experiment.

As to his second stated possibility, very massive monopoles related to SU(5) GUT are still within the realm of possible future experimental discoveries.

Further material about the sine-Gordon doublet Breather and the massive Thirring equation can be found in the book Solitons and Instantons (North-Holland 1982,1987) by R. Rajaraman, who writes:

> "... the doublet or breather solutions ... can be used as input into the WKB method. ... the system is ... equivalent to the massive Thirring model, with the SG soliton state identifiable as a fermion. ... Mass of the quantum soliton ... will consist of a classical term followed by quantum corrections. The energy of the classical soliton ... is ... [eq. 7.3]

E_cl[f_sol] = 8 m^3 / L

The quantum corrections ... to the 'soliton mass' ... is finite as the momentum cut-off goes to infinity and equals (- m / pi). Hence the quantum soliton's mass is [eq. 7.10]

M_sol =(8 m^3 / L) - (m / pi) +O(L).

The mass of the quantum antisoliton will be, by ... symmetry, the same as M_sol. ...

The doublet solutions ... may be quantised by the WKB method. ... we see that the coupling constant (L / m^2) has been replaced by a 'renormalised' coupling constant G ... [eq. 7.24]

G = (L / m^2) / (1 - (L / 8 pi m^2))

... as a result of quantum corrections. ... the same thing had happened to the soliton mass in eq. (7.10). To leading order, we can write [eq. 7.25]

M_sol = (8 m^3 / L) - (m / pi) = 8 m / G

... The doublet masses ... bound-state energy levels ... E = M_N, where ... [eq. 7.28]

M_N = (16 m / G) sin(N G / 16) ; N = 1, 2, ... < 8 pi / G

Formally, the quantisation condition permits all integers N from 1 to oo , but we run out of classical doublet solutions on which these bound states are based when N > 8 pi / G . ... The classical solutions ... bear the same relation to the bound-state wavefunctionals ... that Bohr orbits bear to hydrogen atom wavefunctions. ...

Coleman ... show[ed] explicitly ... the SG theory equivalent to the charge-zero sector of the MT model, provided ... L / 4 pi m^2 = 1 / (1 + g / pi)

...[where in Coleman's work set out above such as his eq. (5.14) , B^2 = L / m^2]...

Coleman ... resurrected Skyrme's conjecture that the quantum soliton of the SG model may be identified with the fermion of the MT model. ... ".

WHAT ABOUT THE NEUTRAL PION?

The quark content of the charged pion is u_d or d_u , both of which are consistent with the sine-Gordon picture.

Experimentally, its mass is 139.57 Mev.

The neutral pion has quark content (u_u + d_d)/sqrt(2) with two components, somewhat different from the sine-Gordon picture, and a mass of 134.96 Mev.

The effective constituent mass of a down valence quark increases (by swapping places with a strange sea quark) by about

DcMdquark = (Ms - Md) (Md/Ms)2 aw V12 =
= 312x0.25x0.253x0.22 Mev = 4.3 Mev.

Similarly, the up quark color force mass increase is about

DcMuquark = (Mc - Mu) (Mu/Mc)2 aw V12 =
= 1777x0.022x0.253x0.22 Mev = 2.2 Mev.

The color force increase for the charged pion DcMpion± = 6.5 Mev.

Since the mass Mpion± = 139.57 Mev is calculated from a color force sine-Gordon soliton state, the mass 139.57 Mev already takes DcMpion± into account.

For pion0 = (u_u + d_d)/ sqrt 2 , the d and _d of the the d_d pair do not swap places with strange sea quarks very often because it is energetically preferential for them both to become a u_u pair. Therefore, from the point of view of calculating DcMpion0, the pion0 should be considered to be only u_u , and DcMpion0 = 2.2+2.2 = 4.4 Mev.

If, as in the nucleon, DeM(pion0-pion±) = -1 Mev, the theoretical estimate is

DM(pion0-pion±) = DcM(pion0-pion±) + DeM(pion0-pion±) =
= 4.4 - 6.5 -1 = -3.1 Mev,
roughly consistent with the experimental value of -4.6 Mev.
