à Monsieur le Professeur Dyson, avec ma profonde reconnaissance

A. Wyler

R. Gilmore. This seems to be the only copy I have. Don't tell Wyler I gave it to you.

F. D.

OPERATIONS OF THE SYMPLECTIC AND SPINOR GROUPS

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1. Introduction

The aim of this paper is the study of actions of the symplectic and spinor groups and an outline of their applications to the n-dimensional harmonic oscillator and to matrix ensembles.[3] The subject of this study was motivated by our research on the conformal group.[9] In the first part we study properties of actions of Lie-groups on manifolds, in particular the trajectories associated to elements of the Lie algebra. We construct explicitly these trajectories in operations of $Sp(n, \mathbb{R})$ and $Spin(n)$; in the case of the transitive action of $Sp(n, \mathbb{R})$ on its symmetric space, they are the geodesics of the invariant metric (chapter 3). The relation with the theory of the n-dimensional harmonic oscillator is given by the property[5]: the Lie algebra of $Sp(n, \mathbb{R})$, is isomorphic to the set of observables, quadratic expressions in the conjugate quantities p, q. The set of elements commuting with the Hamiltonian $\sum_{i=1} p_i p_i + q_i q_i$ is the maximal compact subgroup $SU(n)$ of $Sp(n, \mathbb{R})$; the stability property (chapter 4) of the space of holomorphic functions on $Sp(n, \mathbb{R})/U(n)$ leads to the construction of an orthonormal basis for this space.

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2. Definitions and examples

Let G be a Lie-group: an action of G on a differentiable manifold M^n (not necessarily compact) is a differentiable map

$$\begin{array}{ccc} G \times M^n & \to & M^n \\ g \quad x & \to & g(x) \end{array}$$

such that $g_1(g_2(x)) = g_1 g_2(x)$, $\forall x \in M^n$, $g_1, g_2 \in G$. Therefore an action is given by an homomorphism $G \to \mathrm{Diff}(M^n)$ of G in the group of diffeomorphisms of M^n.

The orbit of a point $x \in M^n$ is the submanifold of M^n:

$$G(x) = \{gx | g \in G\}$$

We consider the following group actions:

1. The symplectic group $Sp(n, \mathbb{R})$ is the set of all $2n \times 2n$ real matrices A which satisfy the relation

$$A^T J A = J \quad \text{where} \quad J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$

a). The linear operation of $Sp(n, \mathbb{R})$ in the euclidean space $\mathbb{R}^{2n}$ is given by:

$$\begin{array}{ccc} Sp(n, \mathbb{R}) \times \mathbb{R}^{2n} & \to & \mathbb{R}^{2n} \\ A \qquad x & \to & Ax \end{array}$$

In particular the tangent space $T_x(M^{2n})$ in each point x of a symplectic manifold M^{2n} has, by definition, the structure group $Sp(n, \mathbb{R})$, invariance group of the form $\sum_{i=1}^{n} dp_i \wedge dq_i$.

b). Let M^r be an homogeneous space of the symplectic group

$$M^r = Sp(n, \mathbb{R})/H$$

where H is a closed subgroup of $Sp(n, \mathbb{R})$; the orbit G_x of each point is M^r. The most important cases are given by $H_1 = \{e\}$, and $H_2 = Sp(n, \mathbb{R}) \cap SO(2n) = U(n)$. The quotient space $Sp(n, \mathbb{R})/U(n)$ is the symmetric space of the symplectic group and the complex imbedding[4]

$$I : Sp(n, \mathbb{R})/U(n) \to \mathbb{C}^{n(n+1)/2}$$

is obtained by the following construction: let $S_n(\mathbb{C})$ be the set of symmetric matrices of order n with complex coefficients and $\Sigma_n(\mathbb{C})$ the subset of $S_n(\mathbb{C})$ given by

$$S_n = X_n + i\, Y_n$$

where $X_n, Y_n \in M_n(\mathbb{R})$ and $Y_n > 0$ (the quadratic form of the symmetric matrix Y_n is positive definite). The operation

$$Sp(n, \mathbb{R}) \times \Sigma_n(\mathbb{C}) \to \Sigma_n(\mathbb{C})$$

is defined by:

$$S_n \to (A_1 S_n + A_2)(A_3 S_n + A_4)^{-1}$$

where $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in Sp(n, \mathbb{R})$, $A_i \in M_n(\mathbb{R})$. To this unbounded realisation of the symmetric space $Sp(n, \mathbb{R})/U(n)$ in $\mathbb{C}^{n(n+1)/2}$ is associated the following bounded realisation: let $B_n(\mathbb{C})$ be the subset of $S_n(\mathbb{C})$ given by the matrices $\{S_n | E > \bar{S}_n S_n\}$. The action of the symplectic group

$$Sp(n, \mathbb{R}) \times B_n(\mathbb{C}) \to B_n(\mathbb{C})$$

is induced by the mapping

$$F : \Sigma_n(\mathbb{C}) \to B_n(\mathbb{C})$$
$$S_n \to (E + i\,S_n)(E - i\,S_n)^{-1}$$

For $g \in Sp(n, \mathbb{R})$ and $B_n \in B_n(\mathbb{C})$

$$g(B_n) = F(g(F^{-1}(B_n)))$$

We obtain the same result as in the study[9] of the conformal group SO(n, 2).

The mapping $F : \Sigma_n(\mathbb{C}) \to B_n(\mathbb{C})$ induces an isomorphism of the isotropy group

$$I(z_0) = \{g \in Sp(n, \mathbb{R}) \,|\, g(z_0) = z_0\}$$

of a point z_0 on the topological boundary $\partial B_n(\mathbb{C}) = \{S_n \,|\, \det(E - \bar{S}_n S) = 0\}$ with the group $GL(\Sigma_n(\mathbb{C}))$ of linear transformations of $\Sigma_n(\mathbb{C})$.

This subgroup of $Sp(n, \mathbb{R})$ is obtained by the restriction of the operation

$$\Sigma_n(\mathbb{C}) \to \Sigma_n(\mathbb{C})$$
$$z \to (A_1 z + A_2)(A_3 z + A_4)^{-1}$$

to the linear action, given by

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ with } A_1 \in GL(n, \mathbb{R})$$

Therefore we have proved

Proposition:

The isotropy group $I(z_0)$ of a point z_0 on $\partial B_n(\mathbb{C})$ is isomorphic to $GL(n, \mathbb{R})$.

2. The second type of operations are induced by the spinor group Spin(k), universal covering group of the orthogonal group SO(k). Let C_k be the Clifford-algebra generated by the elements $e, e_1, \ldots, e_{k-1}$ with the rela-

tions

$$\begin{cases} e_i e_j + e_j e_i = 0 & (i \neq j) \\ e_i^2 = - e \end{cases}$$

The criterion of Hurwitz[7] gives for the order n of the representation of the elements e_i by matrices $A_i \in SO(n)$: if $n = 2^\alpha 16^\beta (2\gamma + 1)$ then $k + 1 = \rho(n) = 2^\alpha + 8\beta$. Let $C^* \subset C_k$ be the multiplicative group of elements with inverse in the Clifford algebra C_k. The spinor group Spin(k) is the identity component of the subgroup of C^* defined by the condition[2]: the linear subspace $< e_1, \ldots, e_{k-1} >$ is invariant under the mapping $y \to xyx^{-1}$ $\forall x \in C^*$. The spin representation, obtained by the left translation $\theta(x) : y \to xy$ restricted to $x \in \mathrm{Spin}(\rho(n))$, induces an action of the spinor group on the Stiefel manifolds $SO(n)/SO(p)$ $(p = 1, 2, \ldots, n)$ by the diffeomorphisms

$$SO(n)/SO(p) \to SO(n)$$

$$y\{SO(p)\} \to \tau(x)y = xy\{SO(p)\}$$

The isotropy group of the point $\pi(e)$, where $\pi : SO(n) \to SO(n)/SO(p)$ denotes the canonical projection, is

$$SO(p) \cap \mathrm{Spin}\,\rho(n)$$

and the orbit of $\pi(e)$ is the Stiefel-manifold $SO(n)/SO(p) \cap \mathrm{Spin}(\rho(n))$.

We obtain as special cases of this construction the classical Hopf fibrations:

<u>$n = 2m,\ p = 2m - 1$:</u> the Clifford algebra C^2 is isomorphic to $\mathbb{C}$ and $\mathrm{Spin}(2) = S^1$. The action is free and the quotient space is $S^{2m-1}/S^1 =$

$P^m(\mathbb{C})$.

<u>$n = 4m$, $p = 4m - 1$</u>: the Clifford algebra C^4 is isomorphic to the algebra of quaternions and Spin(3) is isomorphic to the group S^3 of quaternions of norm 1: the quotient space S^{4m-1}/S^3 is the quaternionic projective space $P^m(H)$.

An interesting application is given by the Hopf fibration $S^{15} \to S^8$ with fiber S^7 constructed with the division algebra of Cayley numbers.[7] This fibration can be constructed as the quotient space of $S^{15} =$ Spin(9)/Spin(7) in an action induced by the spinor representation of Spin(8). This representation is defined in $\mathbb{R}^{16}$ and the action of Spin(8) on the quotient space $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7) \subset \mathbb{R}^{16}$ is given by

$$g\{h\} = \{gh\}$$

where $g \in \mathrm{Spin}(8) \subset \mathrm{Spin}(9)$ and $\{h\}$ is the class of $h \in \mathrm{Spin}(9)$. The only type of orbit is $\mathrm{Spin}(8)/\mathrm{Spin}(7) = S^7$ and the quotient space of the action is S^8; therefore we obtain the fibration $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7) \to S^8$. The natural extension of this construction is given by the spinor representation of Spin(2n +1):

$$\Delta_{2n+1}: \ \mathrm{Spin}(2n+1) \to SO(2^n) \ .$$

Let Δ_{2^n} be the spinor representation of the universal covering group $\mathrm{Spin}(2^n)$ of $SO(2^n)$: the generalisation of Spin(9)/Spin(7) is the quotient space

$$\mathrm{Spin}(2^{n+1})/\mathrm{Spin}(2n+1)$$

where the imbedding $\mathrm{Spin}(2^{n+1}) \supset \mathrm{Spin}(2n+1)$ is given by $\Delta_{2n+1}(SO(2^n))$.

3. The trajectories induced by the group-actions

In the action $G \times M^n \to M^n$ of G on M^n, each element X of the Lie algebra $\mathfrak{g}$ of G induces a vector-field X^* on M^n. Let $\gamma_X(t)$ be the geodesic of the invariant metric on G, tangent to X at the point $e \in G$; the exponential map $\mathfrak{g} \to G$ is defined[4] by $\exp(X) = \gamma_X(1)$ and the trajectory $T_X(q)$ of a point $q \in M^n$ relative to X

$$T_X(q) = \{\gamma_X(t)q \mid -\infty < t < \infty\} .$$

We construct the trajectories of the actions given in chapter 2.

1. Let $sp(n, \mathbb{R})$ be the Lie algebra of the symplectic group:

$$sp(n,\mathbb{R}) = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1^T \end{pmatrix} \middle| \begin{array}{l} X_i \in M_n(\mathbb{R}) \\ X_2, X_3 \text{ symmetric} \end{array} \right\}$$

The trajectory $T_X(q)$, in the case of the action

$$\pi:\ Sp(n,\mathbb{R}) \to Sp(n,\mathbb{R})/U(n) = B_n(\mathbb{C})$$

is given by the geodesic of the invariant metric on $B_n(\mathbb{C})$, with origin q and tangent in this point to the projection $\pi(\exp X)$. We apply therefore the construction of geodesics in symmetric spaces.

The decomposition of the symmetric pair associated to $Sp(n,\mathbb{R})/U(n)$ is

$$sp(n,\mathbb{R}) = u(n) + h(n)$$

where

$$h(n) = \left\{ \begin{pmatrix} g_1 & g_2 \\ g_2 & -g_1 \end{pmatrix} \middle| \begin{array}{l} g_1 \in u(n),\ g_2^T = g_2 \\ g_1 \text{ and } g_2 \text{ imaginary} \end{array} \right\}$$

There exists a riemannian structure g_{ij} on $Sp(n)/U(n)$ such that the left-translations $\ell(a)$ on $Sp(n,\mathbb{R})$ are isometries of the metric g_{ij}.

<u>Proposition:</u> <u>The geodesics of the riemannian structure</u> $\{Sp(n)/U(n), g_{ij}\}$ <u>are the curves</u>

$$f: \ t \to (\ell(g) . \pi)(\exp(tX))$$

<u>where</u> $X \in h(n)$ <u>and</u> $g \in Sp(n, \mathbb{R})$.

The proof of this statement is reduced,[1] as the left action of $g \in Sp(n, \mathbb{R})$ is an isometry, to the study of $\pi(\exp(tX))$ where $X \in h(n)$. The projection

$$\pi: \ Sp(n, \mathbb{R}) \to Sp(n, \mathbb{R})/U(n)$$

is an isometry and by construction the curve $\exp(tX)$ is a geodesic on $Sp(n, \mathbb{R})$; therefore $t \to (\ell(g) . \pi)(\exp(tX))$ is a geodesic on $Sp(n, \mathbb{R})/U(n)$.

We construct an explicit formula for g_{ij} and $|g_{ij}|$. The Bergman kernel[6] is equal to the determinant of the metric tensor, invariant volume element in $B_n(\mathbb{C})$. The action of $Sp(n, \mathbb{R})$ on $B_n(\mathbb{C})$ is given, according to chapter 2, by

$$\begin{aligned} B_n(\mathbb{C}) &\to B_n(\mathbb{C}) \\ z \quad &\to (A_1 z + A_2)(A_3 z + A_4)^{-1} \end{aligned}$$

where $\quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ satisfies $A^T \begin{pmatrix} O & E \\ -E & O \end{pmatrix} A = \begin{pmatrix} O & E \\ -E & O \end{pmatrix}$

Therefore $\begin{cases} A_1{}^T A_3 = A_3{}^T A_1, \ A_2{}^T A_4 = A_4{}^T A_2 \\ -A_3{}^T A_2 + A_1{}^T A_4 = E \end{cases}$

In order to compute the Jacobian of the transformation $z \to A(z)$ and the invariant volume element, we use the fact that the action of $Sp(n, \mathbb{R})$ leaves invariant the boundary of $B_n(\mathbb{C})$, defined by $\det(E - \bar{z}z) = 0$.

Taking account of the dimension $\frac{n(n+1)}{2}$ of $B_n(\mathbb{C})$, we obtain

<u>Proposition:</u> <u>The invariant volume element of</u> $B_n(\mathbb{C})$ <u>is</u>

$$|g_{ij}| = |\det(E - \bar{z}z)|^{-(n+1)}$$

The trajectories of the action of $Sp(n, \mathbb{R})$ on $B_n(\mathbb{C})$ are the geodesics of the metric $g_{ij} = -(n+1)\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ell n|\det(E - \bar{z}z)|$.

By using the map $F: \Sigma_n(\mathbb{C}) \to B_n(\mathbb{C})$ (chapter 2) we reduce the study of the trajectories in the action of $Sp(n, \mathbb{R})$ on the unbounded domain

$$\Sigma_n(\mathbb{C}) = \{X_n + i\, Y_n | Y_n > 0\}$$

to the preceding case (bounded domain $B_n(\mathbb{C})$). The mapping F commutes namely with the action of $Sp(n, \mathbb{R})$; in the diagram

$$\begin{array}{ccc} B_n(\mathbb{C}) & \xrightarrow{g \,\in\, Sp(n, \mathbb{R})} & B_n(\mathbb{C}) \\ F\uparrow & & \uparrow F \\ \Sigma_n(\mathbb{C}) & \xrightarrow{g \,\in\, Sp(n, \mathbb{R})} & \Sigma_n(\mathbb{C}) \end{array}$$

$gF = Fg$ for all $g \in Sp(n, \mathbb{R})$ and therefore the image by F^{-1} of a geodesic in $B_n(\mathbb{C})$ is a geodesic in $\Sigma_n(\mathbb{C})$.

2. In the action of $Spin(\rho(n))$ on $S^{n-1} = SO(n)/SO(n-1)$, constructed in chapter 2, the trajectories of the elements A_i in the Lie algebra $spin(\rho(n)) \cong so(\rho(n))$ are the integral curves of the vector-fields $X_i = A_i x$ on S^{n-1}. These vector-fields are orthogonal because

$$\begin{aligned} < X_i, X_j > &= < A_i x, A_j x > = < A_i^2 x, A_i A_j x > \\ &= - < x, A_i A_j x > \end{aligned}$$

and $$< X_i, X_j > = < A_j A_i x, A_j^2 x > = < A_i A_j x, x >$$
$$= - < X_i, X_j >$$

Therefore $< X_i, X_j > = 0 \quad (i \neq j)$.

Proposition: The Lie algebra generated by the infinitesimal transformations of the vector-fields $X_i = A_i x$ is isomorphic to $so(\rho(n))$.

Proof: The vector-field $X_i = A_i x$ is defined[2] by $\sum_{r=1}^{\infty} A^i{}_r \frac{\partial}{\partial x_r}$ where

$$A^i{}_r = \sum_{\ell=1}^{n} a^i{}_{r\ell} \, x_\ell$$

The Lie-bracket $[X_i, X_j]$ is

$$[X_i, X_j] = \left(A^j{}_r \frac{\partial A^i{}_s}{\partial x_r} - A^i{}_r \frac{\partial A^j{}_s}{\partial x_r} \right) \frac{\partial}{\partial x_s}$$
$$= 2(A_i A_j)_{r\ell} \, x_\ell \quad .$$

because of the relation $A_i A_j = - A_j A_i \; (i \neq j)$.

In the Clifford algebra $C_{\rho(n)}$ generated by the elements $A_1, \ldots, A_{\rho(n)}$ the subspace $M_2 = < A_i A_j > \; (i \neq j)$ of dimension $\frac{\rho(n)}{2}(\rho(n) - 1)$ is isomorphic to $so(\rho(n))$.

We apply this result to the integrability of the distribution of the $\rho(n)$ vector-fields $X_1, \ldots, X_{\rho(n)}$ on S^{n-1} (existence of submanifolds $M^{\rho(n)}$ of S^{n-1}, such that in each point $x \in M^{\rho(n)}$, $T_x(M^{\rho(n)}) = < X_1, \ldots, X_{\rho(n)} >$). The condition for integrability is

$$[X_i, X_j] \in < X_1, \ldots, X_{\rho(n)} > \qquad \forall_i \neq j$$

Since $[X_i, X_j] = 2(A_i A_j)x$, the condition is satisfied in the case of the Hopf-fibrations S^{2n-1}/S^1 and S^{4n-1}/S^3, the fibers (S^1 and S^3) defining the tangent submanifolds of the distribution. In the general case, the quotient space

$$S^{n-1} \to S^{n-1}/\mathrm{Spin}(\rho(n))$$

induces on S^{n-1} a foliation with singularities. The set Σ of singularities is given by

$$\Sigma = \left\{ x \in \mathbb{R}^n \,\middle|\, \left(\sum_{i,k} \lambda_{ik} A_i A_k \right) x = 0 \right\}$$

and in the complement $S^n - \Sigma$, we obtain a foliation, given by a covering $\{U_i, f_i\}$ of $S^n - \Sigma$ such that the coordinate transformations

$$(f_j \circ f_i)^{-1} \colon \; f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$$

are decomposed as

$$\begin{aligned} (x, y) &\to (h^1_{ij}(x), h^2_{ij}(y)) \\ \mathbb{R}^n &\to \mathbb{R}^{\rho(n)} \times \mathbb{R}^{n-\rho(n)}. \end{aligned}$$

The fibers are the integral manifolds of the given distribution $X_1, \ldots, X_{\rho(n)}$.

4. Applications

The Hamiltonian[5] of the n-dimensional harmonic oscillator is a quadratic form

$$H(p, q) = \sum a_{ij} \, p_i q_j$$

in the conjugate quantities p_i, q_j and the set of observables which are quadratic expressions in these variables is the Lie algebra of the symplectic group $Sp(n, \mathbb{R})$. The maximal compact subgroup $SU(n) = Sp(n, \mathbb{R}) \cap SO(2n)$ of $Sp(n, \mathbb{R})$ is given by the set of elements commuting with the Hamiltonian

$$\sum_{i=1}^{n} (p_i p_i + q_i q_i)$$

In the quotient space $Sp(n,\mathbb{R})/U(n)$, the relation between the conjugate quantities p, q is given by the complex structure of $M^N = Sp(n,\mathbb{R})/U(n)$, imbedded in $\mathbb{C}^{n(n+1)/2}$ (chapter 2).

We introduce the complex coordinates

$$z_k = x_k + i\, y_k \qquad (k = 1, \ldots, N = \frac{n(n+1)}{2})$$

and the tangent space $T(M^N)$ of the open submanifold M^N has the basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_N}, \frac{\partial}{\partial y_N}$.

The complex structure is defined by the endomorphism

$$J: \; T(M^N) \to T(M^N)$$

where $J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$, $J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$.

The connection of the complex structure J with the invariant metric on $Sp(n,\mathbb{R})/U(n)$ (chapter 3) is: a) the endomorphism J is an isometry of the tangent space; b) J is invariant under parallel translation.[4] The following stability-property leads to the geometrical quantization for the harmonic oscillator by introducing the complex structure J in the phase space. The space $H(Sp(n,\mathbb{R})/U(n))$ of holomorphic functions on the symmetric space M^N is an invariant irreducible subspace of the action of $Sp(n,\mathbb{R})$. We construct an orthonormal basis of $H(Sp(n,\mathbb{R})/U(n))$; the Silov boundary[6] $\Sigma(Sp(n,\mathbb{R})/U(n))$ is the manifold of all symmetric matrices of order n. Let $S \in \Sigma\,(Sp(n,\mathbb{R})/U(n))$; we consider the elements s_{ij} of the matrix S as the components of a vector $s \in \mathbb{C}^{n(n+1)/2}$. Let U be an unitary matrix and

$$T = USU' \in \Sigma\,(Sp(n,\mathbb{R})/U(n))$$

The matrix U induces therefore a linear transformation

$$U: \mathbb{C}^{\frac{n(n+1)}{2}} \to \mathbb{C}^{\frac{n(n+1)}{2}}$$

$$s \to u(s) = t$$

Let $P^f(\mathbb{C}^{n(n+1)/2})$ be the space of homogeneous polynomials of degree f in the variables $\{s_{ij}\}$. U induces a linear transformation

$$U^f: P^f\left(\mathbb{C}^{\frac{n(n+1)}{2}}\right) \to P^f\left(\mathbb{C}^{\frac{n(n+1)}{2}}\right)$$

The basis of the invariant subspaces of U^f give an orthonormal system in the space $L^2(\Sigma(Sp(n, \mathbb{R})/U(n)))$ of square integrable functions on the Silov boundary. As a consequence of the isometry

$$L^2(\Sigma(Sp(n, \mathbb{R})/U(n))) \to H(B^n(\mathbb{C}))$$

$$f(w) \qquad \to \int P_n(z, w) f(w) dw$$

where
$$P_n(z, w) = \frac{1}{V(\Sigma)} \; \frac{\det(E - zz^T)^n}{\det(z-w)^{2n}}$$

we obtain an orthonormal basis on $H(B^n(\mathbb{C}))$.

Remark 1:

Let $SU(n, p)$ be an extension of the maximal compact subgroup $SU(n)$ of $Sp(n, \mathbb{R})$. The homogeneous space $SU(n, p)/SU(n, p-1)$ is realized in $\mathbb{C}^{n+p}$ by the hyperboloid $H^{n,p}$

$$\sum_{i=1}^{n} z_i \bar{z}_i - \sum_{k=1}^{p} z_{n+k} \bar{z}_{n+k} = 1$$

The Lie-algebra of invariant vector fields on $H^{n,p}$ is a subalgebra of $su(n, p)$ where

$$su(n,p) = \left\{ \begin{pmatrix} A_1 & A_2 \\ {A_2}^T & A_3 \end{pmatrix} \middle| \begin{array}{l} A_1 = -\bar{A}_1,\ A_3 = -\bar{A}_3 \\ Tr(A_1) + Tr(A_3) = 0 \end{array} \right\}$$

The following curves define trajectories of the action of $SU(n,p)$ on $H^{n,p}$

$$\begin{cases} z_j(t) = a_j e^{it} & (j = 1, \dots, n) \\ z_k(r) = b_k e^{it} & (k = 1, \dots, p) \end{cases} \qquad \text{with } \Sigma {a_i}^2 - \Sigma {b_k}^2 = 1$$

In the case $p = 1$, a symplectic structure (with linear action of $Sp(n, \mathbb{R})$ on the tangent space) is defined by the fundamental form $\sum_{j=1}^{n} da_j \wedge d\bar{a}_j$ where

$$a_j = \frac{1}{\sqrt{2}} (p_j + i\, q_j)$$

<u>Remark 2:</u>

One can apply the preceding construction of an orthonormal system in the symmetric space of the symplectic group to the ensembles of hermitian matrices considered by Dyson[3] in "Algebraic structure of symmetry groups and ensembles in quantum mechanics."

Let $M_n(F)$ be the set of matrices of order n with coefficients in the field F (either $\mathbb{R}$, $\mathbb{C}$ or H); we consider subsets of $M_n(F)$ invariant under the action of a group G operating in F^n by the representation:

$$\rho\colon\ G \to GL(F^n)$$

Dyson studies the two cases of invariance:

a) $M_n^f(F) = \{S \in M_n(F) | S\rho(g) = \rho(g)S \quad \forall g \in G\}$

The matrices are then called formally invariant.

b) $M_n^p(F) = \left\{ S \in M_n(F) \,\middle|\, \begin{array}{ll} S\rho(u) = \rho(u)S, & \text{if } \rho(u) \text{ unitary} \\ S\rho(a) = \rho(a)\bar{S}^T, & \text{if } \rho(a) \text{ antiunitary} \end{array} \right\}$

Then the matrices are called physically invariant; the two definitions are equivalent if the matrices S are hermitian.

Dyson obtains the classification result:

Theorem: An irreducible set of matrices in $M_n(F)$, invariant under G in the physical sense is the set of all self-dual matrices in F. In the case of an unitary representation, the set of matrices invariant under G, is an orthogonal, unitary or symplectic group.

The ensembles of matrices formally invariant under G are the group-manifolds $SO(n)$, $SU(n)$, and $Sp(n,\mathbb{R})$ with the probability distribution induced by the Haar-measure on this group; for the symplectic group we apply the construction of the invariant metric on $Sp(n,\mathbb{R})/U(n)$ (chapter 2). The volume element

$$|g_{ij}| = |\det(E - z\bar{z})|^{-n+1}$$

induces by the canonical projection

$$i: \; Sp(n,\mathbb{R}) \to Sp(n,\mathbb{R})/U(n)$$

the Haar-measure on $Sp(n,\mathbb{R})$, uniquely defined up to a multiplicative constant.

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THE COMPLEX LIGHT CONE, SYMMETRIC SPACE OF THE CONFORMAL GROUP.

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INTRODUCTION

The mathematical structure and the physical interpretation of the conformal group, invariance group of Maxwell equations, are studied in this paper, as a detailed analysis of our publication[16] which has been the object of several papers.[13,14,17]

The first part gives the structure of the conformal group in a metric of signature (p, q) and a proof of the relation between the conformal group and the isometry group of a compact manifold.

In the Minkowski metric (n-1, 1), we construct the bounded realization $D^n \subset \mathbb{C}^n$ of the symmetric space of the conformal group, the unbounded realization being the complex light cone $T^n = \mathbb{R}^n + i\, V^n$ where

$$V^n = \{ y \in \mathbb{R}^n \mid y_1 > 0,\ y_1^2 - \ldots - y_n^2 > 0 \}$$

This explicit construction allows to define the Bergman metric, the invariant differential operators and their elementary solutions (Green functions) in the bounded realization D^n of $SO(n, 2)/SO(n) \times SO(2)$. Their domain of definition is the Cartesian product of D^n with the Silov boundary Q^n. We prove that Q^n is the quotient space $C(M^n)/P(M^n)$ of the conformal group by the Poincaré group $P(M^n)$ and give several applications (representations of $C(M^n)$ induced by $P(M^n)$, eigenvalues of Casimir operators in the Lie algebra of $C(M^n)$).

The construction of the Green functions in D^n is extended to the non-scalar case (tensor- and spinor-fields) and the value of the structure constant α is obtained as coefficient of the Green function of the Dirac equation in D^5.

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1. The Structure of the Conformal Group

A conformal mapping $f: M^n \to M^n$ of a pseudo-Riemannian manifold M^n with metric tensor g_{ij} is a differentiable homeomorphism of M^n which leaves invariant the isotropic cones:

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}\ dx^i\ dx^j = 0$$

If we introduce the induced map f_* for tensors on M^n, then $f_*(g_{ij}) = \lambda^2 g_{ij}$, where $\lambda \neq 0$. The conformal group $C(M^n)$ is the group of these transformations, and is an extension of the group $I(M^n)$ of isometries, which keep the metric tensor g_{ij} invariant.

Example 1:

Let $M^{p,q}$ be the generalized Minkowski space with the metric of signature (p, q) $ds^2 = dx_1^2 + \ldots + dx_p^2 - dx_{p+1}^2 - \ldots - dx_{p+q}^2$ (where $pq \neq 0$). The isometry group $I(M^{p,q})$ is the Poincaré group $P(M^{p,q})$, semi-direct product of the group of translations $T^{p,q}$ and of the generalized Lorentz-group:

$$SO(p,q) = \{A \in M^{p+q}(\mathbb{R}) \mid A^T E_{p,q} A = E_{p,q}\}$$

where
$$E^{p,q} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ 0 & & & & & -1 \end{pmatrix} \begin{matrix} \}\ p \\ \\ \}\ q \end{matrix}$$

$M^{p,q}$ is an homogeneous space of the Poincaré group $P(M^{p,q})$; the isotropy group of this action is $SO(p,q)$ and therefore

$$M^{p,q} = P(M^{p,q}) / SO(p,q).$$

The following transformations of $C(M^{p,q})$ are not contained in $P(M^{p,q})$:

1. the dilatation $D(a,\lambda)$ of center a and factor λ:

$$x_i \to \lambda (x_i - a_i)$$

2. the inversion $I(b)$ of center b:

$$x_i \to \frac{x_i - b_i}{N^2(x, b)}$$

where

$$N^2(x, b) = (x_1 - b_1)^2 + \dots + (x_p - b_p)^2 - (x_{p+1} - b_{p+1})^2 - \dots - (x_{p+q} - b_{p+q})^2$$

The inversion $I(b)$ is a differentiable homeomorphism in the complement $M^{p,q} - C(b)$ of the isotropic cone $C(b)$ of center b.

According to Liouville's theorem, the conformal transformations are obtained by composition of the rotations, translations, dilatations, and inversions. In order to obtain the structure of the Lie-group $C(M^{p,q})$, we introduce the geometry of spheres in the Minkowski space $M^{p,q}$.

Theorem:

The set $S(p, q)$ of all hyperboloids in $M^{p,q}$ is an homogeneous space of the group $S(M^{p,q})$ of this geometry, which is isomorphic to the group $SO(p+2, q+1)$.

Proof:

The proof is a generalization of[15]:

the equation of an hyperboloid in $M^{p,q}$ is

$$x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2 + 2\sum_{i=1}^{p+q} x_i a_i + b = 0.$$

The radius r is given by:

$$r^2 = a_1^2 + \ldots + a_p^2 - a_{p+1}^2 - \ldots - a_{p+q}^2 + b$$

and the introduction of homogeneous coordinates

$$a_i = \frac{\xi_i}{\xi_{p+q+3}} \ (i=1, \ldots, p+q) \ b = \frac{\xi_{p+q+1}}{\xi_{p+q+3}}, \quad r = \frac{\xi_{p+q+2}}{\xi_{p+q+3}}$$

gives, after a linear transformation

$$\xi_1^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \ldots - \xi_{p+q}^2 + \xi_{p+q+1}^2 + \xi_{p+q+2}^2 - \xi_{p+q+3}^2 = 0.$$

The linear group of this form is $SO(p+2, q+1)$.

We give the relation between the conformal group and the group of the geometry of spheres; let $M^{p-1, q}$ be the pseudo-riemannian space imbedded in $M^{p, q}$ by $x_1 = 0$. We associate to each isotropic cone $ds^2 = 0$ of center x in $M^{p, q}$ the hyperboloid, intersection of this cone with the hyperplane $x_1 = 0$. This isotropic projection from the set $C(p, q)$ of isotropic cones in $M^{p, q}$, homogeneous space under the action of the conformal group $C(M^{p, q})$, into the set $S(p-1, q)$ of spheres in $M^{p-1, q}$ induces an isomorphism of the conformal group $C(M^{p, q})$ with the group $S(M^{p-1, q}) \cong SO(p+1, q+1)$.

Example 2:

In order to illustrate the difference of structure with the preceding construction, we study the conformal group $C(M^n)$ of a compact riemannian manifold M^n and we prove that, with the exception of the homology sphere, $C(M^n)$ is isomorphic to the group $I(M^n)$ of isometries; we give a new proof of the following theorem, more in line with the representation theory of groups.

Theorem:

If the conformal group $C(M^n)$ *of a compact manifold* M^n *is not isomorphic to the group* $I(M^n)$ *of isometries, then the Betti numbers* $b_p(M^n)$, *dimensions of the homology groups of* M^n, *are zero for* $p = 1, \ldots, n-1$.

Proof:

Let g_{ij} be the riemannian metric on M^n; we define the Laplace operator in the space of p-forms and representations of the groups $C(M^n)$ and $I(M^n)$ in the spaces of harmonic forms: the differential $d\alpha$ of a p-form $\alpha = a_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \in \Lambda^p(M^n)$ is given by $(da)_{i_1 \ldots i_{p+1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{p+1}}$, the codifferential $\delta\alpha = (\delta a)_{i_1 \ldots i_{p-1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{p-1}}$ with $(\delta a)_{i_1 \ldots i_{p-1}} =$ $= (-1)^{np+n+1} (* d * a)_{i_1 \ldots i_{p-1}}$ where $*\alpha$ is the adjoint[6] of α.

The Laplace operator is then defined by

$$\Delta = d\delta + \delta d : \Lambda^p(M^n) \to \Lambda^p(M^n)$$

and the space $H_p(M^n)$ of harmonic p-forms as

$$H_p(M^n) = \{\alpha \in \Lambda^p(M^n) \mid \Delta\alpha = 0\}.$$

According to Hodge-theorem,[6] the dimension of $H_p(M^n)$ is given by the p-th Betti-number $b_p(M^n)$. On the other hand, because of the invariance of the Laplace equation $\Delta\alpha = 0$ in the operation of the conformal group $C(M^n)$ and of the isometry group $I(M^n)$, we obtain representations of degree $b_p(M^n)$ of these groups:

$$R_{C(M^n)} : C(M^n) \to GL\, H_p(M^n)$$

$$g \to R(g)$$

with $R(g)\ (\alpha(x)) = \alpha(g^{-1}(x))$

for $\alpha \in H^p(M^n)$ and $x \in M^n$,

and the same formula for the induced representation of $I(M^n) \subset C(M^n)$. If there exists an element $g \in C(M^n)$, not contained in $I(M^n)$, with $g(g_{ij}) = \lambda^2 g_{ij}$ $(\lambda \neq 1)$ then[6] there is no harmonic form of degree p, $0 < p < n$, except the trivial one, and therefore $b_i(M^n) = 0$ for $i = 1, \ldots, n-1$.

This method of demonstration leads us to the study of invariant differential operators and the representations of $C(M^n)$ in their spaces of solutions, in the case $C(M^n) \neq I(M^n)$, as considered in the example of the generalized Minkowski space.

2. The Bounded Realisation of the Complex Light Cone.

Let $T^{p,q} = \mathbb{R}^{p+q} + i\,V^{p,q}$ $(pq \neq 0)$ with

$$V^{p,q} = \{\, y \in \mathbb{R}^{p+q} \mid y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_{p+q}^2 > 0 \,\}$$

$$y_1 > 0, \ldots, y_p > 0$$

be the complex cone associated to the metric of signature (p, q); this domain of $\mathbb{C}^{p+q}$ has the complex structure of the imbedding $T^{p,q} \to \mathbb{C}^{p+q}$. The group operating transitively on $T^{p,q}$, generated by the translations on the real part R^{p+q}, the rotations leaving $V^{p,q}$ invariant and the inversions

$$z_i \to \frac{z_i}{z_1^2 + \ldots + z_p^2 - z_{p+1}^2 - \ldots - z_{p+q}^2}$$

is, according to the results of the first section, isomorphic to $SO(p+1, q+1)$. Our aim is to construct a bounded realization of the space $T^{p,q}$ in $\mathbb{C}^{p+q}$; this means an holomorphic map $F : T^{p,q} \to \mathbb{C}^{p+q}$ such that $F(T^{p,q})$ is contained in a sphere of $\mathbb{C}^{p+q}$ and such that F commutes with the group operation on $T^{p,q}$. We apply the theorem of E. Cartan[7] which shows the special structure of the conformal group in a space M^n of signature $(n-1, 1)$.

<u>Theorem:</u>

<u>The tubular domain</u> $T^{p,q} = \mathbb{R}^{p+q} + i\, V^{p,q}$ <u>admits a bounded realisation if and only if</u> $q = 1$.

The construction is divided into 2 parts:

a) the bounded realisation of the symmetric space $D^n = SO(n,2) / SO(n) \times SO(2)$

b) the holomorphic map $F : D^n \to T^n$

a) We consider an homogeneous space of the linear action of the group $SO(n,2)$ in the complex projective space $P_{n+2}(\mathbb{C})$, namely the hyperboloid

$$H_{n,2} = \{ x \in P_{n+2}(\mathbb{C}) \mid x^T E_{n,2}\, x = 0 \}$$

and construct a stereographic projection[12] from $P_{n+2}(\mathbb{C})$ in $\mathbb{C}^n$ such that the linear operation of $SO(n,2)$ on $H_{n,2}$ is transformed into a group of homographic transformations of a bounded domain of $\mathbb{C}^n$.

We introduce the non-homogeneous coordinates

$$\begin{cases} z_1 = (x_{n+1} + i\, x_{n+2})^{-1} (x_1 + i\, x_2) \\ z_2 = (x_{n+1} + i\, x_{n+2})^{-1} (x_1 - i\, x_2) \\ z_j = (x_{n+1} + i\, x_{n+2})^{-1}\, x_j \quad (3 \le j \le n) \end{cases}$$

The subset Σ_{n+2} of the hyperboloid $H_{n,2}$, given by the inequality $\bar{x}^T E_{n,2}\, x > 0$ is transformed into the bounded domain of $\mathbb{C}^n$

$$\begin{cases} 1 + \left|\sum_{i=1}^{n} z_i z_i\right|^2 - 2\sum_{i=1}^{n} z_i \bar{z}_i > 0 \\ \left|\sum_{i=1}^{n} z_i z_i\right| < 1 \end{cases}$$

The isotropy group of the linear action of SO(n, 2) on the subset $\Sigma_{n,2}$ of $H_{n,2}$ is $SO(n) \times SO(2)$; the mapping $S : \Sigma_{n,2} \to D^n$ given by the preceding formulas induces an isomorphism

$$S_* \; : \; SO(n, 2) \;\to\; A(D^n)$$

of the group SO(n, 2) onto the group $A(D^n)$ of holomorphic transformations of D^n; therefore

$$D^n \;=\; SO(n, 2) \;/\; SO(n) \;\times\; SO(2) \,.$$

b) We construct in the second part of the proof the holomorphic mapping $F : D^n \to T^n = \mathbb{R}^n + i\, V^n$ of the bounded realisation D^n in the complex light cone T^n.

Let $S : \Sigma_{n,2} \to D^n$ be the stereographic projection constructed in a): by a linear transformation of variables, the equation of the subset $\Sigma_{n,2}$ of the hyperboloid $H_{n,2}$ becomes

$$Y^T M_{n,2}\; Y = 0 \,, \qquad \bar{Y}^T M_{n,2}\; Y > 0$$

where

$$M_{n,2} \;=\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & -E_{n,1} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We introduce the coordinates (with $y_1 \neq 0$)

$$z_1 = y_1^{-1} y_{n+2}, \quad z_j = y_1^{-1} y_j \quad \text{for } j = 2, \ldots, n.$$

The equations of $\Sigma_{n,2}$ imply:

$$(\operatorname{Im} z_1)^2 - (\operatorname{Im} z_2)^2 - \ldots - (\operatorname{Im} z_n)^2 > 0 .$$

If we consider the connected component $\operatorname{Im} z_1 > 0$ of this set, we obtain the complex light cone $T^n = \mathbb{R}^n + i V^n$ and the map $T : \Sigma^{n,2} \to T^n$. The composition-map $F = T S^{-1} : D^n \to T^n$ gives therefore the holomorphic transformation of T^n in a bounded domain of $\mathbb{C}^n$ with the property: the mapping $F : D^n \to T^n$ induces an isomorphism $F_* : I(z) \to L(T^n)$ of the isotropy group $I(z)$ of a point z (on the boundary of D^n) with the group $L(T^n)$ of linear transformations of T^n, isomorphic to the Poincaré group $P(M^n)$: a linear transformation of T^n leaves invariant a point with the homogeneous coordinates $y_1 \neq 0$, $y_j = 0$ ($j = 2, \ldots, n$) which is a boundary point of D^n.

The explicit form of the bounded realisation allows to compute the functional determinant of the transformation $F : D^n \to T^n$ and the non-linear action of $SO(n, 2)$ on D^n [10,12]; the Jacobian is obtained from the invariance condition on the domain T^n, the only polynomial invariant under the linear action of $P(M^n)$ being $(z_1^2 - z_2^2 - \ldots - z_n^2)^k$: the dimension implies $k=n$. The explicit computation of the functional matrix $\frac{\partial z_i}{\partial y_k}$ gives the Jacobian $\left| \frac{\partial z_i}{\partial y_k} \right| = 2^n (z_1^2 - \ldots - z_n^2)^{-n}$. For the action of a group element $A \in SO(n,2)$ on D^n, induced from the linear action on the hyperboloid $\Sigma_{n,2} \subset P_{n+2}(\mathbb{C})$, we decompose [8] the matrix A with $(n+2)$ rows and $(n+2)$ columns in the following way:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where the matrices A_i $(i=1,\dots,4)$ are respectively 2×2, $n\times 2$, $2\times n$ and $n\times n$. Then the map

$$\begin{aligned} D^n &\to D \\ A : z &\to A(z) \end{aligned}$$

is given by:

$$A(z) = \left[\left(\begin{pmatrix} zz' + 1 \\ i(zz' - 1)\end{pmatrix} \frac{A_1^T}{2} + z A_2^T\right)\begin{pmatrix} 1 \\ i\end{pmatrix}\right]^{-1}\left[\begin{pmatrix} zz' + 1 \\ i(zz' - 1)\end{pmatrix} \frac{A_3}{2} + z A_4^T\right]$$

The image in the map $F : D^n \to T^n$ of the real part $(\mathbb{R}^n, \{0\})$ of the complex light cone $T^n = \mathbb{R}^n + i V^n$ is given by

$$Q_n = \left\{\xi = xe^{i\theta} \in \mathbb{C}^n \;\middle|\; x = (x_1, \dots, x_n),\ x_1^2 + \dots + x_n^2 = 1\right\}$$

and the isotropy group $SO(n) \times SO(2)$ of an interior point of D^n operates transitively on Q^n, cartesian product of two spheres of dimension $n-1$ and 1. The subset Q^n (Silov boundary) of the topological boundary ∂D^n of D^n is also an homogeneous space of the action of $SO(n, 2)$ with isotropy group $P(M^n)$ (chapter 7)

Remark:

In the case $n = 4$, the complex light-cone $T^n = \mathbb{R}^n + i V^n$ has a special bounded realisation in a space of matrices. Namely, the Lie algebra of the conformal group $SO(4, 2)$ is isomorphic to the Lie algebra of the group $SU(2, 2)$ associated to the hermitian form

$$z_1\bar{z}_1 + z_2\bar{z}_2 - z_3\bar{z}_3 - z_4\bar{z}_4$$

Therefore, the bounded realisation is a domain of the first type in the classification of E. Cartan.[8]

$$D^{2,2} = \{ z \in M_2(\mathbb{C}) \mid E_2 - \bar{z}^T z > 0 \}$$

(where the symbol $A > 0$ for a symmetric matrix means that the associated form is positive-definite) and the action of the group of automorphisms

$$SU(2,2) = \{ M \in M_4(\mathbb{C}) \mid \bar{M}^T E_{2,2} M = E_{2,2} \}$$

is given by $z \to (Az + B)(Cz + D)^{-1}$ with the decomposition $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

3. <u>Construction of the Metric Tensor on D^n.</u>

The bounded realisation D^n of the symmetric space $SO(n,2)/SO(2)$ in $\mathbb{C}^n$, given in the previous chapter, allows us to define on D^n an invariant metric tensor[7]: we give in this section an explicit construction of this metric, which we apply to the Green function on D^n.[10] Let γ_{ij} and g_{ij} be the metric tensors on $T^n = \mathbb{R}^n + iV^n$ and D^n; the mapping $F : D^n \to T^n$ gives

$$\left| (\gamma_{ij}) \right| = J(F) \left| g_{ij} \right|$$

where $J(F) = (1 + |zz'|^2 - 2\bar{z}z')^{-n}$ is the Jacobian of F. Because of the pseudo-euclidean structure of the metric on T^n, we obtain for the invariant volume element on D^n

$$| g_{ij} | = c(1 + |zz'|^2 - 2\bar{z}z')^{-n}$$

where c is a constant factor.

Remarks:

1. This formula can be obtained by using the fact that the action of $SO(n, 2)$ on D^n, given in the previous section, leaves invariant the boundary of D^n, defined by the polynomial equation $1 + |zz'|^2 - 2\bar{z}z' = 0$. Therefore the invariant volume element has the form $|g_{ij}| = c\,(1 + |zz'|^2 - 2\bar{z}z')^{\lambda}$.

2) An analytic construction[7] of the invariant volume element is given by the Bergman kernel. Let $\varphi_1(z)$, $\varphi_2(z)$, ... be an orthonormal basis in the Hilbert space $H(D^n)$ of holomorphic functions on D^n, with the scalar product induced by the euclidean measure on $\mathbb{R}^{2n}$; the series $\sum_{p=1}^{\infty} \varphi_p(z)\,\overline{\varphi_p(z)}$ converges uniformly on each closed subset of the Cartesian product $D^n \times D^n$ and defines an holomorphic function $K(z, \bar{z})$ independent of the choice of basis in $H(D^n)$: we have the following transformation law:

Proposition:

Let $g : z \to gz$ be the mapping of D^n given by $g \in SO(n, 2)$, then $K(z, \bar{z}) = K(gz, g\bar{z})\,|j_g(z)|^2$.

Proof:

The sequence $\varphi_1(gz), \ldots, \varphi_p(gz)$ does not give an orthonormal basis in $H(D^n)$ because the euclidean volume element is not invariant in the action of $SO(n,2)$; but $\{\varphi_p(gz)\; j_g(z)\}$ is orthonormal because

$$\delta_{pq} = \int_{D^n} \varphi_p(z)\;\overline{\varphi_q(z)}\;\prod_{i=1}^{n} dz_i\;d\bar{z}_i$$

$$= \int_{D^n} \varphi_p(gz)\;\varphi_q(gz)\,|j_g|^2\;\prod_{i=1}^{n} dz_i\;d\bar{z}_i$$

The sum of the series $\sum \varphi_p(z)\, \varphi_p(\bar{z})$ is independent of the basis and we obtain the transformation law

$$K(z, \bar{z}) = \sum \varphi_p(gz)\varphi_q(gz)\, j_g(z) j_g(\bar{z}) = |j_g(z)|^2 \; K(gz, \overline{gz}).$$

Since the Bergman kernel $K(z, \bar{z})$ and the volume element $|g_{ij}|$ have the same transformation law in the action of $SO(n, 2)$, they differ only by a constant proportionality factor

$$K(z, \bar{z}) = c_1 |g_{ij}| = c_2^{-1} (1 + |zz'|^2 - 2\bar{z}z')^{-n}$$

This factor c_2 is given by the euclidean volume $V(D^n)$ of the bounded domain D^n, equal to $\frac{\pi^n}{2^{n-1} n!}$, in the case $\lambda = 0$ of the formula.[8]

$$\int_{D^n} (1 + |zz'|^2 - 2\bar{z}z')^{\lambda} \prod_{i=1}^{n} dz_i\, d\bar{z}_i = \frac{\pi^n}{2^{n-1}} \; \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+n)} \cdot \frac{1}{(2\lambda+n)}$$

3) In [16,17] we apply a different transformation formula

$$G(z) = G(g(z))\, |j_g(z)|^4$$

which is associated to the volume element of the Riemannian structure

$$G(z)^{\frac{1}{2}} = c\, K(z, \bar{z})$$

The transformation law is a consequence of the formula $K(z, \bar{z}) = K(gz, \overline{gz})\, j_g^2$ and of the relation

$$|j_g(z)|^2 = \frac{\partial (u_1, v_1, \ldots, u_n, v_n)}{\partial (x_1, y_1, \ldots, x_n, y_n)}$$

between the complex Jacobian $j_g(z) = \dfrac{\partial (w_1, \ldots, w_n)}{\partial (z_1, \ldots, z_n)}$ and the real Jacobian of the transformation $z \rightarrow g(z)$.

The invariant metric tensor g_{ij} is defined by:

$$g_{ij} = \frac{\partial^2}{\partial z_i \, \partial \overline{z_j}} \ln (1 + |zz'|^2 - 2\bar{z}z')^{-n}$$

We compute g_{ij} explicitly, in order to apply it to the induced metric of the imbedding $i : D^{n-1} \to D^n$ given by $z_n = 0$. We obtain:

$$g_{ij} = -n \frac{\partial}{\partial z_i} \left(\frac{\partial}{\partial \bar{z}_j} \ln \left[1 + (\Sigma z_k z_k)(\Sigma \bar{z}_\ell \overline{z_\ell}) - 2 \Sigma z_p \bar{z}_p \right] \right)$$

$$= -n \frac{\partial}{\partial z_i} \left(\frac{2\bar{z}_j (\Sigma z_k z_k) - 2 z_j}{1 + |zz'|^2 - 2zz'} \right)$$

$$= 2n \left[\frac{(1 + |zz'|^2 - 2z\bar{z}')(\bar{z}_j z_i - \delta_{ij}) - (\bar{z}_j (\Sigma z_k z_k) - 2z_j)(2z_i \Sigma z_k z_k - 2\bar{z}_j)}{(1 + |zz'|^2 - 2zz')^2} \right]$$

The inclusion map $I : D^{n-1} \to D^n$ given by the restriction $(z_1, \ldots, z_{n-1}) \to (z_1, \ldots, z_{n-1}, 0)$ induces a metric $I_*(g_{ij})$ on D^n and we obtain, according to the previous formula

$$\left. (g_{\mu,\nu}(z_1, \ldots, z_n) \right|_{z_n = 0} = \frac{n}{n-1} \left(\begin{array}{c|c} (\frac{1}{n-1}) g_{ij}(z_1, \ldots, z_{n-1}) & 0 \\ \hline 0 & 1 + |zz'|^2 - 2z\bar{z}' \end{array} \right)$$

$$\mu, \nu = 1, \ldots, n$$

In particular, the volume elements on $D^{n-1} \subset D^n$ are related by

$$\left| g_{\mu,\nu}(z_1, \ldots, z_n) \right|_{z_n = 0} = \frac{n^n}{(n-1)^{n-1}} \left| g_{ij}(z_1, \ldots, z_{n-1}) \right|$$

4. Unitary Representations of the Conformal Group (Discrete Series).

The previous construction of the invariant metric leads to a scalar product in the space $H(D^n)$ of holomorphic functions on D^n which is invariant in the discrete series of representations.

For any integer p, we construct the representation

$$R_p : SO(n,2) \to GL(H(D^n))$$

$$g \to R_p(g)\, f(z) = f(g^{-1}z)\, j_g^{\,p}(z)$$

where $j_g(z)$ is the Jacobian of the mapping $g : z \to gz$.

In order to define the invariant scalar product, we apply the transformation formula of the volume element $|g_{ij}|$, proportional to the Bergman kernel (see Chapter 3):

$$|g_{ij}(z)| = |g_{ij}(gz)| \cdot |j_g(z)|^2$$

Therefore the scalar product

$$< f_1, f_2 >_p = \int_{D^n} f_1(z)\, \overline{f_2(z)}\; K^{-p}(z, \bar{z})\; dz_1 \ldots d\bar{z}_n$$

is invariant in the transformation $R_p(g)$ and the representation $R_p : SO(n,2) \to GL(H(D^n))$ is unitary because:

$$<R_p(g)\, f_1,\; R_p(g) f_2 >_p = \int_{D^n} f_1(g^{-1}z)\, f_2(g^{-1}z)\; K^{-p}(gz, g\bar{z})\, |j_g(z)|^{2p}\, dz_1 \ldots d\bar{z}_n$$

$$= \int_{D^n} f_1(z)\, \overline{f_2(z)}\; K^{-p}(z, \bar{z})\; dz_1 \ldots d\bar{z}_n$$

$$= < f_1, f_2 >_p$$

Let $\varphi_1(z), \ldots, \varphi_k(z), \ldots$ be an orthonormal basis of $H(D^n)$, (with the measure induced by the euclidean structure; then $\{ \varphi_K(z)\; K^{-\frac{p}{2}} \}$ is an orthonormal basis in the scalar product $< f_1, f_2 >_p$. The matrix of the operator $R_p(g)$ is given by

$$a_p^{k\ell}(g) = \langle R_p(g)\,\varphi_k(z),\ \varphi_\ell(z)\rangle$$

$$= \int_{D^n} \varphi_k(g^{-1}z)\ \varphi_\ell(z)\ K^{-p}(z,\bar{z})\ j_g^p(z) \prod dz_i\, d\bar{z}_i$$

According to the result of Bargmann,[1] the coefficients $a_p^{k\ell}(g)$ of the representation R_p are square integrable on the group manifold SO(n, 2):

$$\int_{SO(n,2)} \left| a_p^{k\ell}(g)\right|^2 dg < \infty$$

dg being the invariant measure on SO(n, 2). This property is characteristic of the discrete series of holomorphic representations of SO(n, 2), according to the result.[1]

<u>Theorem:</u>

<u>Let</u> $R : SO(p,q) \to U(H)$ <u>be a unitary representation of the group</u> SO(p, q) (with $pq \neq 0$) <u>in an Hilbert space H.</u> <u>If the coefficients</u> $\langle R(g)\varphi_i,\ \varphi_k\rangle$ <u>of the representation</u> R, <u>where</u> φ_i <u>is an orthornal basis of</u> H, <u>are square integrable on</u> SO(p, q), <u>then</u> $p = 2$ <u>or</u> $q = 2$ <u>and</u> R <u>is equivalent to a representation of the discrete series.</u>

5. <u>Elementary Solutions of the Laplace Operators in M^n and D^n.</u>

The holomorphic map $F : D^n \to T^n$, constructed in section 2, gives the following relation between the Green functions on D^n and M^n.

Let $\Box_{1,n-1}$ be the Laplace operator in M^n and $H_{1,n-1}$ the space of harmonic functions:

$$H_{1,n-1} = \{\ \varphi \mid \Box_{1,n-1}\,\varphi = 0\ \}$$

The invariance of the equation $\square_{1,n-1}\varphi = 0$ in the action of the conformal group $C(M^n) \cong SO(n+2, 2)$ induces a representation of $SO(n,2)$

$$R_n : SO(n,2) \to GL(H_{1,n-1})$$

$$g \quad \to R_n(g) : \varphi(x) \to \varphi(g^{-1}(x))$$

We construct first the elementary solution E_n of $\square_{1,n-1}$ which satisfies $\square_{1,n-1} E_n = \delta$, in the case $n = 2k$. To the Dirac distribution $\delta(x)$ in $\mathbb{R}^1$ corresponds by the mapping:

$$f : M^n \to \mathbb{R}$$

$$(x_n, \dots, x_n) \to d = x_1^2 - \dots - x_n^2$$

the distribution $H(\varepsilon) = f_*(\delta(\varepsilon))$ with

$$\text{supp}\,(H(\varepsilon)) = \{ (x_1, \dots, x_n) \mid d = \varepsilon \}$$

In the same way, to the derivative of order p of δ_ε corresponds by the mapping $f : M^n \to \mathbb{R}$ a distribution $H^{(p)}(\varepsilon)$.

The elementary solution E_n of $\square_{1,n-1}$ is given by[5]:

$$E_n = \frac{1}{(4\pi)^{\frac{n-2}{2}}} \lim_{\varepsilon = 0} H_\varepsilon^{(\frac{n-4}{2})}$$

Let us consider the holomorphic function $(z_1^2 - \dots - z_n^2)^{\frac{n-2}{2}}$ in the complex light cone $T^n = \mathbb{R}^n + iV^n$ and its symmetric $T^{n(-)}$ associated to the cone

$$V^{n(-)} = \{ y \in \mathbb{R}^n \mid y_1^2 - \dots - y_n^2 - \dots - y_n^2 > 0,\ y_1 < 0 \};$$

the boundary values are defined by

$$\left((x_1 + i0)^2 - x_2^2 - \dots - x_n^2\right)^{-(\frac{n-2}{2})} = \lim_{\varepsilon_1 \to 0} \left((x_1 + i\varepsilon_1)^2 - (x_2^2 - \dots - x_n^2)\right)^{-(\frac{n-2}{2})}$$

The difference of the boundary values in the complex light cone V^n and its symmetric $V^{n(-)}$ gives an elementary solution of the wave equation

$$0 = \Box_{1,n-1}\left[\left((x_1+i0)^2 - x_2^2 - \dots - x_n^2\right)^{-\frac{n-2}{2}} - \left((x_1-i0)^2 - x_2^2 - \dots - x_n^2\right)^{-\frac{n-2}{2}}\right]$$

We apply now the mapping defined in section 2,

$$F : T^n = \mathbb{R}^n + iV^n \to D^n = SO(n,2)/SO(n) \times SO(2)$$

which induces an isomorphism

$$F_* : I(z_o) \cong P(M^n)$$

of the isotropy group of a point z_o of the boundary of D^n with the group $P(M^n)$ of linear transformations of T^n.

This gives the relation between the Laplace operators on M^n and D^n and therefore between the elementary solutions of these operators.

Theorem:

The image of the invariant operator $\Box_{1,n-1}$ of the mapping $F : D^n \to T^n$ is the Laplace operator of the invariant metric on D^n.

Proof:

We apply the following result[7] to the group $G = SO(n,2)$; let G be a connected semi simple Lie group with Lie algebra $\mathfrak{g}$ and invariant form B. The differential operator $\Sigma\, g_{ij} X_i X_j$ (where $\{X_i\}$ is a basis of $\mathfrak{g}$ and $g_{ij} = B(X_i, X_j)$), called the Casimir operator of G, corresponds in the mapping $G \to G/K$ of G on its symmetric space G/K to the invariant Laplace operator on G/K. The Casimir operator of the Poincaré group is therefore the wave operator $\Box_{1,n-1}$ and the Casimir operator of $SO(n,2)$

is the Laplace operator

$$\Delta = \frac{1}{\sqrt{g}} \left(\sum_k \frac{\partial}{\partial x_k} \left(\sum_i g_{ik} \frac{\partial}{\partial x_i} \right) \right)$$

of the invariant metric (g_{ij}) defined on $D^n = SO(n,2)/SO(n) \times SO(2)$ in section 4. The result follows now from the property for the map $F: D^n \to T^n$ of inducing the isomorphism $I(z) \cong P(M^n)$. Therefore $F_*(\Box_{1,n-1}) = \Delta$. A corollary of this result is the relation induced by F of the elementary solutions of the wave operator $\Box_{1,n-1}$ and of the Laplace operator on $D^n = SO(n,2)/SO(n) \times SO(2)$. We construct now the Green function of Δ; in chapter 2, we have obtained that the image in the mapping $F: T^n \to D^n$ of the subset of

$$T^n = \{ z = x + iy \mid x \in \mathbb{R}^n, \; y \in V^n \}$$

defined by $y = 0$ is the Silov boundary

$$Q_n = \{ \xi = x e^{i\theta} \mid \sum_{i=1}^{n} x_i^2 = 1 \}$$

contained in the topological boundary of D^n, and invariant in the action of the group $SO(n,2)$.

Q^n is homeomorphic to the cartesian product of the two spheres S^{n-1} and S^1 and has therefore a measure dg invariant under the action of $SO(n) \times SO(2)$, defined by the tensor product of the riemannian measures on S^{n-1} and S^1.

Q^n has the following properties, characteristic of the Silov boundary.[8]

a) Let $f(z)$ be an holomorphic function in D^n such that its absolute value $|f(z)|$ is bounded on the closure $\overline{D^n}$; then the maximum in $\overline{D^n}$ of $|f(z)|$ is reached on Q^n.

b) Let $q \in Q^n$ be an arbitrary point of the Silov boundary; then there exists an holomorphic function $f(z)$ in D^n such that $\underset{z \in D^n}{\mathrm{Max}} |f(z)|$ is reached on Q^n.

Therefore, for any holomorphic function $f(z)$ on D^n, we have

$$f(z) = \int_{Q^n} f(\xi) \; dq_z(\xi)$$

where dq_z is the measure on Q^n, invariant under the action of the isotropy group of z, isomorphic to $SO(n) \times SO(2)$. For the complex light cone T^n, the relation[10] between the values of an holomorphic function on the Silov boundary and in T^n is:

$$f(k_1 + i k_2) = \frac{1}{A(S^{n-1})} \int_{R^n} \frac{f(p) <k, k> dp_1 \dots dp_n}{[(\|k_1 - p_1\| - \|k_2\|)^2 + 4 <k_1 - p_1, k_2>^2]^{\frac{n}{2}}}$$

where the scalar product $<k_1, k_2>$ is given by the Minkowski metric and $A(S^{n-1})$ is the area of the $(n-1)$ dimensional unit-sphere.

In the transformation $F : D^n \to T^n$, the image of an holomorphic function is holomorphic (since F is analytic) and the Silov boundary $\mathbb{R}^n$ of T^n is transformed in Q^n:

$$f(z) = \int_{Q^n} f(\xi) \; P_n(z, \xi) \; d\xi$$

where the Poisson kernel $P_n(z, \xi)$ is given by

$$P_n(z, \xi) = \frac{\left(V(D^n)\right)^{\frac{1}{2}}}{V(Q^n)} \; \frac{(1 + |zz'|^2 - 2\bar{z}z')^{\frac{n}{2}}}{|(z - \xi)(z - \xi)'|^n}$$

the measure $V(Q^n)$ of the Silov boundary being $\left(\Gamma(\frac{n}{2}) \right)^{-1} \; 2\pi^{\frac{n}{2}+1}$.

Proposition:

$P_n(z,\xi)$ is an harmonic function of z.

Proof:

$P_n(z,\xi)$ is an holomorphic function of z and as a consequence of the Kaehlerian structure of the metric[6] $\Delta_z P_n(z,\xi) = 0$.

By construction, $P_n(z,\xi)$ is (as function of $\xi \in Q^n$) a measure on Q^n, invariant under the action of the isotropy group of the point z, isomorphic to $SO(n) \times SO(2)$.

Remark:

The relation of the Poisson kernel $P_n(z,\xi)$ with the Bergman kernel $K_n(z,\bar{z}) = V(D_n)^{-1}(1+|zz'|^2 - 2\bar{z}z')^{-n}$ is the following: we introduce an orthonormal basis $\{\varphi_i(z)\}$ in the Hilbert space $H(D^n)$ of holomorphic functions on D^n:

According to chapter 3, $K_n(z,\bar{w}) = \sum_{\nu=0}^{\infty} \varphi_\nu(z)\,\overline{\varphi_\nu(w)}$; the sequence $\{a_i^{-\frac{1}{2}}\,\varphi_\nu(\xi)\}$ where

$$\int_{Q^n} \varphi_i(\xi)\,\varphi_i(\xi)\,d\xi = a_i$$

is orthonormal in the Hilbert space $L_2(Q^n)$ and the Poisson kernel is given by the quotient of series

$$P_n(z,\xi) = \frac{|\Sigma\, a_i^{-1}\varphi_i(z)\,\varphi_j(\xi)|}{\Sigma\, a_i\,\varphi_i(z)\,\varphi_j(\bar{z})}$$

6. Special Properties of the Elementary Solutions in D^n (n even).

The Silov boundary of $D^n = SO(n,2)/SO(n) \times SO(2)$, given by $Q^n = S^{n-1} \times S^1 = \{\zeta \in \mathbb{C}^n \mid \zeta = xe^{i\theta},\ xx' = 1\}$ has the following property:

Theorem:

Q^n is a complex manifold if and only if n is even. In this case, the complex structure is invariant in the transitive action of the groups $SO(n) \times SO(2)$ and $SO(n,2)$.

Proof:

For n even, the structure of complex manifold is given by the identification map

$$M^n \to M^n/\Gamma$$

where $M^n = \mathbb{C}^n - \{0\}$ and Γ is the discrete group of holomorphic transformations of M^n generated by the map

$$f : \mathbb{C}^n - \{0\} \to \mathbb{C}^n - \{0\}$$
$$z_i \to 2z_i$$

This group acts freely on M^n and therefore its quotient space is a complex manifold, isomorphic to $S^{n-1} \times S^1$.

For $n = 2k+1$, the dimension of the Silov boundary is odd and therefore Q^n is not complex.

The second part of the theorem on the invariance of the complex structure of Q^{2k} under the group actions $SO(n) \times SO(2)$ and $SO(n, 2)$ follows from the linearity of the action of $SO(n) \times SO(2)$ and from the analyticity of the action of $SO(n, 2)$.

Remark:

Although $S^{2k} \times S^1$ is a complex manifold, it is not a Kaehler-manifold; namely its first Betti number is one for $k > 1$, whereas the first Betti number of a compact Kaehler manifold is even.[6]

We obtain as a corollary of the preceding theorem.

Corollary:

Let $R_1 : SO(n, 2) \to GL(H(S^{n-1} \times S^1)$ be the representation of $SO(n, 2)$ in the space of holomorphic functions on $S^{n-1} \times S^1$. This representation is equivalent to a representation of $SO(n, 2)$ in the space $H(D^n)$ of holomorphic functions on $D^n = SO(n, 2)/SO(n) \times SO(2)$.

Proof:

The Poisson integral induces an isomorphism $I : H(S^{n-1} \times S^1) \cong H(D^n)$ by associating to each holomorphic function $g \in H(S^{n-1} \times S^1)$ the function $f \in H(D^n)$ given by $f(z) = \int_{Q^n} P_n(z, \xi)\ g(\xi)\ d\xi$. This mapping I_* commutes with the representation of the group $SO(n, 2)$ on $H(D^n)$ and $H(S^{n-1} \times S^1)$; hence the equivalence of the representations.

The Poisson kernel of the domain D^n, associated to the conformal group $C(M^4) \cong SO(4, 2)$ has the following property.

Theorem:

The Poisson kernel $P_4(z, \xi)$ is an harmonic function with respect to both variables $z \in D^4$ and $\xi \in Q^4$; the Laplace operator on $D^4 \times Q^4$ being constructed from the tensor product of the invariant metrics.

Proof:

It is already known that the Poisson kernel $P_n(z, \xi)$ is harmonic with respect to z and is an invariant measure on $Q^n = S^{n-1} \times S^1$; $Q^4 = S^3 \times S^1$ has the structure of a Lie group, given by the cartesian product of the group S^3 of quaternions of norm 1 with the group S^1 of complex numbers of norm 1. Since the Poisson kernel $P_4(z, \xi)$ is an invariant measure on Q^4, we can apply the following result: on a compact Lie group, a necessary and sufficient condition for a form to be harmonic is the invariance of this form[6]; therefore, $P_4(z,\xi)$ is harmonic on $D^4 \times Q^4$.

7. <u>The Quotient Space of the Conformal Group by the Poincaré Group.</u>

The application of the quotient space $C(M^n)/P(M^n)$ is the following: the conformal group $C(M^4)$, extension of the 10-dimensional Poincaré group $P(M^4)$, induces 5 additional conservation identities, associated to the generators of the Lie algebra of $C(M^4)$, not contained in the Lie algebra of $P(M^4)$ (chapter 8); it is therefore important to study the representations of the Lie algebra generated by these elements in the Hilbert space $L^2(C(M^4)/P(M^4))$. We have the result:

<u>Theorem:</u>

<u>The group</u> $SO(n, 2)$ <u>of analytic transformations of the symmetric space</u> D^n <u>operates transitively on the Silov boundary</u> $S^{n-1} \times S^1$ of D^n; <u>the isotropy group of this action is isomorphic to the Poincaré group</u> $P(M^n)$.

<u>Proof:</u>

1) We prove first the invariance of the boundary Q^n under the action of $SO(n, 2)$; since the subgroup $SO(n) \times SO(2)$ of $SO(n, 2)$ operates transitively on Q^n, there exists for each pair $(p_1, p_2) \in Q^n \times Q^n$ a group element $g \in SO(n, 2)$ such that $g(p_1) = p_2$. The invariance of Q^n under this action follows from the properties a) and b) of the Silov boundary, given in the preceding chapter:

Let g be an element of $SO(n, 2)$ and q a point of Q^n such that $g(q) \notin Q^n$. According to the property b), there exists a function f on

D^n with $\underset{D^n}{\text{Max}}(|f|)$ in q. The function $f(gz)$ is analytic and bounded in $\overline{D}^n$, but reaches the maximum of its norm in $g(q)$, outside of Q^n, in contradiction with the property a): therefore $g(q)$ is necessarily in Q^n and SO(n, 2) leaves the Silov boundary invariant.

2) We prove now that the isotropy group of the action of SO(n, 2) on Q^n is isomorphic to the Poincaré group: We use the Cayley transformation $F: D^n \to T^n$ (chapter 2) which allows to construct the isotropy group in a point z_o of the boundary; F induces an isomorphism $F^*: I(z_o) \cong \overline{L}(T^n)$ of this isotropy group with the group of linear transformations of the complex light cone T^n, which is isomorphic to the Poincaré group $P(M^n)$; as a consequence of the transitivity of the action of SO(n, 2) on Q^n, we obtain: $Q^n = SO(n, 2)/P(M^n)$.

We give an application of this result to the space $F_n(T^4)$ of functions defined on the cartesian product of n copies of the cone T_4, and invariant under the action of the Poincaré group:

$$f(gz_1, \ldots, gz_n) = f(z_1, \ldots, z_n)$$

for all $z_i \in T^4$ and $g \in P(M^4)$. Because of this invariance, the functions are defined on the product

$$\prod_{i=1}^{n} (SO(4, 2)/P(M^4))_i$$

which is the Silov boundary of the space $\prod_{i=1}^{n} (D^4)_i$. Namely, according to the previous theorem $Q^4 = SO(4, 2)/P(M^4)$ and the Silov boundary of the cartesian product of spaces is given by the product of the Silov boundaries.

We obtain therefore: the map $F: \prod_{i=1}^{n} (D^4)_i \to \prod_{i=1}^{n} (T^4)$ induces the isomorphism

$$F_n(T^4) \cong \bigotimes_{i=1}^{n} (H(Q^4))_i \cong \bigotimes_{i=1}^{n} (H(D^4))_i$$

Another application of the quotient space $C(M^n)/P(M^n)$ is the study of the induced representation of $C(M^n)$ by a representation of $P(M^n)$: we give first the general definition of an induced representation.[11]

Definition:

Let K *be a closed subgroup of the Lie group* G *and let* $\rho : K \to U(V)$ *be a unitary representation of* K *in the Hilbert space* V. *Let* W *be the vector space of functions* $f : G \to V$ *such that* $f(kg) = \rho(k) f(g)$ *for all* $k \in K$ *and* $g \in G$. *The representation*

$$I(\rho) : G \to U(W)$$

$$g \to I_\rho(g) : f(x) \to f(xg)$$

is a unitary representation of G, *called the induced representation of* G *by* ρ.

We take in the application $G = SO(n, 2)$, $K = P(M^n)$ and according to the preceding section, $G/K = Q^n$. The Silov boundary Q^n of D^n being a compact homogeneous space, the invariant measure dv on Q^n is uniquely defined; we obtain a unitary representation of $SO(n, 2)$ in the space of functions of Q^n in $U(V)$. On the other hand, the functions f on Q^n are boundary values of holomorphic functions g on $D^n = SO(n, 2)/SO(n) \times SO(2)$ given by the Poisson formula

$$g(z) = \int_{Q^n} P_n(z, \xi)\, f(\xi)\, d\xi$$

Therefore the induced representation of $C(M^n)$ is equivalent to a representation of $SO(n, 2)$ in $U(V)$.

8. The Conservation Identities of the Conformal Group and the Eigenvalues of Casimir Operators.

The generators of the Lie algebra $SO(4, 2)$ of the conformal group $C(M^4)$ have the following commutation relations.[15] Let P_i, M_{ij} be the infinitesimal translations and rotations $P_i = \frac{\partial}{\partial x_i}$, $M_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$; then $[P_i, P_j] = 0$, $[P_i, M_{jk}] = g_{ij} P_k - g_{ij} P_j$ and $[M_{ij}, M_{k\ell}] = -g_{ik} M_{j\ell}$ $g_{jk} M_{i\ell} + g_{i\ell} M_{jk} - g_{j\ell} M_{ik}$. The generator corresponding to the dilatation

$$x_i \to \lambda x_i$$

is $D = x^i \frac{\partial}{\partial x_i} + 1$ whereas the operator L_i of the conformal transformation

$$x_i \to \frac{x_i - a^i <x, x>}{1 - 2 <a, x> + <a>^2 <x>^2}$$

($<a, x>$ meaning the Lorentz scalar product) is given by[3].

$$L_i = \left(- <x, x> \frac{\partial}{\partial x^i} + 2 x_i (x^j \frac{\partial}{\partial x_j} + 1) \right)$$

The commutation relations are

$$[D, L_i] = -L_i, \ [D, P_i] = P_i$$

$$[P_i, L_j] = 2 L_{ij} - g_{ij} D$$

$$[M_{ij}, L_i] = g_{ii} L_j$$

$$[M_{ij}, D] = 0$$

Because of the symmetry between the elements L_i and P_i, we obtain the result:

Proposition:

The Lie algebra of the conformal group contains two subalgebras isomorphic to the Lie algebra of the Poincaré group, namely the algebras generated by $\{M_{ij}, P_k\}$ and $\{M_{ij}, L_k\}$.

As corollary we obtain that the Casimir operator of the Lie algebra generated by $\{M_{ij}, L_k\}$ is given by $L_1^2 - L_2^2 - L_3^2 - L_4^2$. In order to apply the results of the chapter 7 on the quotient space $C(M^4)/P(M^4)$ to the eigenvalues of the operator $L_1^2 - L_2^2 - L_3^2 - L_4^2$, we give the conservation identities[15] associated to these generators of the Lie algebra of $C(M^4)$. Let T_{ij} be the energy-momentum tensor of the field invariant under the action of the conformal group; for example, in the case of the Maxwell field $H_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$,

$$T_{ij} = \sum_{k=1}^{4} H_{ik} H_{kj} + \frac{1}{4}\delta_{ij} \sum_{j,\ell=1}^{4} H_{j\ell}^2$$

The conservation identities associated to the elements D and L_i are:

$$\sum_i \frac{\partial}{\partial x_i} \Big(\sum_j x_j T_{ji}\Big) = 0; \quad \sum_i \frac{\partial}{\partial x_i} \Big(2x_j \sum_k x_k T_{ki} - T_{ji} \sum_k x_k^2\Big) = 0$$

We can now compute the eigenvalues of the Casimir operator $L_1^2 - L_2^2 - L_3^2 - L_4^2$; the homogeneous space $C(M^4)/P(M^4)$ is the Silov boundary $Q^4 = S^3 \times S^1$, and therefore the Casimir operator $L_1^2 - L_2^2 - L_3^2 - L_4^2$ is the invariant Laplace operator on the compact manifold Q^4; as Q^4 is the cartesian product of two manifolds we apply the general case of the product $M_1 \times M_2$ with Riemannian metric $g_1 \otimes g_2$. We have the result[2]

<u>Proposition:</u>

<u>The spectrum of the Laplace operator of</u> $M_1 \times M_2$ <u>is given by</u>

$$\mathrm{Spec}(M_1 \times M_2) = \{ \lambda + \mu \mid \lambda \in \mathrm{Spec}(M_1, g_1),\ \mu \in \mathrm{Spec}(M_2, g_2) \}$$

In chapter 10, we obtain the set of eigenfunctions of the Laplace operator on S^n and by applying the result for n=1 and n=3, we obtain

<u>Corollary:</u>

<u>The spectrum of the Casimir operator</u> $L_1^2 - L_2^2 - L_3^2 - L_4^2$ <u>is given by</u>

$$\mathrm{Spec}(Q^4) = \{ k(k+1) \mid k \geq 0 \}.$$

9. <u>Conformal Invariance of Free Fields.</u>

The Klein-Gordon equation of a scalar field is $(\Box - m^2)\varphi(x) = 0$. The invariance group of this equation is the Poincaré group $P(M^4)$; if $m = 0$, the invariance group is the conformal group $C(M^4) \cong SO(4, 2)$. The Fourier transform $a(k)$ of $\varphi(x)$ is $\neq 0$ on the mass-hyperboloid[9]

$$H_m = \{ k \mid \langle k, k\rangle = m^2 \}$$

and we have the representation

$$\varphi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int a(k)\, e^{i\langle k, x\rangle}\, dV_m(k)$$

$dV_m(k)$ being the invariant measure on the hyperboloid H_m.

As special case, the elementary solution $S_m(x)$ of $(\Box - m^2)\varphi(x) = 0$ is given by

$$a_m(k) = \begin{cases} +1 & \text{for } k \in \{ H_m \wedge \{ k_1 > 0 \} \} \\ -1 & \text{for } k \in \{ H_m \wedge \{ k_1 < 0 \} \} \end{cases}$$

According to chapter 1, the set of all mass hyperboloids of M^4 is an homogeneous space of the conformal group SO(5, 2) of the Minkowski space of signature (1, 4); as the Fourier transform of the elementary solution $S_m(x)$ is constant on the hyperboloid H_m, we define the space of functions $L(\Sigma_5)$ on the $\Sigma_5 = SO(5,2)/SO(4,2)$ of all hyperboloids of M^4.

The stereographic projection of the hyperboloid SO(5, 2) /SO(4, 2) into the bounded domain $D^5 = SO(5,2)/SO(2)$ (chapter 2) induces an isomorphism of $L(\Sigma_5)$ into the space $H(D^5)$ of holomorphic functions on D^5.

The connection between the space of solutions of differential operators in M^n, invariant under the action of the conformal group $C(M^n) \cong SO(n,2)$, and the space $H(D^n)$ is given by the following result:

We have obtained in chapter 5: the elementary solutions E_n of the wave equation in M^n,

$$\Box_n E_n = \delta$$

are constructed as boundary values of holomorphic functions in the unbounded realisation

$$T^n = \mathbb{R}^n + i V^n$$

of the symmetric space $D^n = SO(n,2)/SO(n) \times SO(2)$.

Theorem:

The representation of $C(M^n) \cong SO(n,2)$ in the space of solutions $\{\varphi \mid \Box_n \varphi = 0\}$ of the wave equation, is equivalent to the representation of $SO(n,2)$ in the space $H(D^n)$ of holomorphic functions on D^n.

Proof:

The Fourier-Laplace transform g(z) of $\varphi(x)$,

$f(z) = \int \varphi(x)\, e^{i<z,x>}\, dx_1 \ldots dx_n$ is an holomorphic function in $T^n = \mathbb{R}^n + i V^n$ and the action of $C(M^n)$ in M^n induces equivalent representations of SO(n, 2) in $\{ \varphi \mid \Box_n \varphi = 0 \}$ and in the space $H(T^n)$ of holomorphic functions on T^n. By introducing the map $F : T^n \to D^n$ of the complex light cone T^n in the bounded domain D^n (see chapter 2), we obtain the equivalence of the representations in $\{ \varphi \mid \Box_n \varphi = 0 \}$ and $H(D^n)$.

We apply the result of chapter 6 on the equivalence of the representation in $H(D^n)$ with the representation of SO(n, 2) in $H(Q^n)$, in the case of $n = 2k$ and we obtain the corollary.

Corollary:

For n even, the representation of SO(n, 2) in $\{ \varphi \mid \Box_n \varphi = 0 \}$ is reducible and the decomposition in invariant subspaces is given by the action of SO(n, 2) on the compact space $Q^n = S^{n-1} \times S^1$.

Remark:

The special structure of the solutions of the wave equation $\Box_n \varphi = 0$ for n even (space dimension odd) is characterized by the Huyghens principle. Another approach to this connection with the set of hyperboloids in M^n is given in [4].

10. The Tensor Algebra of $H(D^n)$.

We apply now the construction of the tensor algebra on the vector space of solutions of

$$\Box_n \varphi = 0 \quad (n = 4 \text{ and } 5)$$

Because of the invariance of the wave equation under the action of the conformal group $C(M^n) \cong SO(n, 2)$, we obtain, according to the preceding section, representations of $SO(n, 2)$ in the space $H(D^n)$ of holomorphic functions on D^n.

We construct in this chapter an orthonormal basis $\{e_i\}$ of $H(D^n)$ with scalar product $<f_1, f_2> = \int_{D^n} |f_1 f_2| \prod_{i=1}^{n} dz_i\, d\bar{z}_i$. We give first the definition[9] of the tensor power over $H(D^n)$

<u>Definition:</u>

The tensor power of order m $(H(D^n))^{\otimes m}$ of $H(D^n)$ is the Hilbert space of multilinear m-forms on $H(D^n)$ such that

$$\sum_{(i_1, \ldots, i_m)} |\varphi(e_{i_1}, \ldots, e_{i_m})|^2 < \infty$$

The orthonormal basis $\{e_i\}$ in $H(D^n)$ induces an orthonormal basis on $H(D^n)^{\otimes m}$, namely $\{e_{i_1} \otimes \ldots \otimes e_{i_m}\}$ where $(e_{i_1} \otimes \ldots \otimes e_{i_m})$ is the m-form defined by $(e_{i_1} \otimes \ldots \otimes e_{i_m})(f_1, \ldots, f_m) = \prod_{k=1}^{m} <e_{i_k}, f_k>$ with $f_i \in H(D^n)$.

The tensor algebra on $H(D^n)$ is defined as the direct sum

$$T(H(D^n)) = \bigoplus_{m=0}^{\infty} (H(D^n))^{\otimes m}$$

which contains as subspaces the Grassmann algebra $G(H(D^n))$ and the symmetric algebra $S(H(D^n))$. In order to construct orthonormal bases in these spaces, we give an explicit construction of such a basis on $H(D^n)$. Let $L^2(S^{n-1})$ be the Hilbert space of square integrable functions on the sphere $S^{n-1} = SO(n)/SO(n-1)$. We decompose the representation

$$T : SO(n) \rightarrow GL(L^2(S^{n-1}))$$

given by $T(g)(f(x)) = f(g^{-1}x)$ for $g \in SO(n)$ into irreducible subspaces.

Let $P_k(\mathbb{R}^n)$ be the vector space of homogeneous polynomials of degree k in n variables; by considering the restriction $S^{n-1} \subset \mathbb{R}^n$ given by

$$f(\rho\, x) = \rho^k f(x) \qquad (x \in S^{n-1})$$

we obtain that $P_k(\mathbb{R}^n)$ is an invariant subspace[8] of $L^2(S^{n-1})$ with dimension

$$d(P_k(\mathbb{R}^n)) = (k!)^{-1} n(n+1) \ldots (n+k-1).$$

The representation $\mathbb{R} : SO(n) \rightarrow GL(P_k(\mathbb{R}^n))$ is reducible; let $H_k(\mathbb{R}^n)$ be the vector space of homogeneous harmonic polynomials of degree k on $\mathbb{R}^n$; the Laplacian Δ on $S^{n-1} \subset \mathbb{R}^n$ gives:

$$H_k(\mathbb{R}^n) = \{ f \in P_k(\mathbb{R}^n) \mid (\Delta + \lambda) f = 0, \quad \lambda = k(n+k-2) \}$$

and one obtains[8]

<u>Theorem:</u>

<u>The orthogonal decomposition of</u> $P_k(\mathbb{R}^n)$ <u>into invariant subspaces under the action of</u> $SO(n)$ <u>is</u>

$$P_k(\mathbb{R}^n) = H_k \oplus H_{k-2} \oplus \ldots \oplus H_{k-2[\frac{k}{2}]}$$

<u>where</u> $[\frac{k}{2}]$ <u>is the integer part of</u> $\frac{k}{2}$.

Let $\pi_{k-2\ell}(x)$ be the projection of the vector $x^{(k)} \in P^k(\mathbb{R}^n)$ with components

$$\frac{\ell!}{i_1! \ldots i_n!} \; x_1^{i_1} \ldots x_n^{i_n} \; (\text{where } \sum_{p=1}^{n} i_p = k)$$

in the subspace $H_{k-2\ell}$ of harmonic polynomials:

The vector $\pi_{k-2\ell}(x)$ has the components $(xx')^{\ell} f^i_{k-2\ell}(x)$ and the functions $g^i_{k-2\ell} = (c_{k-2\ell})^{\frac{1}{2}} f^i_{k-2\ell}(x)$ with $< f^i_j , f^i_j > = c_j$ form an orthonormal system on S^{n-1}.

The cartesian product $S^{n-1} \times S^1$ is the Silov boundary Q^n of D^n (chapter 4) and the functions $\frac{(\xi \xi')^{\ell}}{\sqrt{\pi}} g^i_{k-2\ell}(\xi)$, where $\xi = e^{i\theta} x$, give an orthonormal basis on $L^2(Q^n)$. We have proved, in section 5, that the mapping

$$\begin{aligned} H(D^n) &\to L^2(Q^n) \\ f(z) &\to f|_{Q^n} \end{aligned}$$

induces an isometry of the Hilbert space $H(D^n)$ of holomorphic functions on D^n with the space $L^2(Q^n)$. Therefore we obtain:

Theorem:

The system $\{ e_J \} = \{ (zz')^{\ell} \; g^i_{k-2\ell}(z) \}$ is an orthonormal basis of $H(D^n)$.

An orthonormal basis in the tensor algebra $T(H(D^n))$ is given by introducing the elements $\{ e_{j_1} \otimes \ldots \otimes e_{j_m} \}$ of $(H(D^n))^{\otimes m}$

11. Exterior Forms and Spinor Forms on D^n.

In order to extend the theory to the non-scalar case, we define fields of spin $\neq 0$ on the bounded realisation D^n of the complex light cone $T^n = \mathbb{R}^n + i V^n$. First we construct the space of exterior forms on the complex manifold D^n: let $T_x(D^n)$ be the tangent space in $x \in D^n$ and $(T_x(D^n))^*$ its dual space, with basis $dz_1, \ldots, dz_n, d\bar{z}_1, \ldots, d\bar{z}_n$.

The elements $dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \ldots \wedge d\bar{z}_{i_q}$ form a basis of the space $\bigwedge_x^{p,q}(D^n)$ of bihomogeneous forms of degree (p, q) on D^n. The exterior algebra on $T_x(D^n)$ is the direct sum

$$\bigwedge T_x(D^n) = \bigoplus_{p,q} \bigwedge^{p,q}(T_x(D^n))$$

The natural extension of the holomorphic functions on D^n, defined by $\frac{\partial f}{\partial \bar{z}_j} = 0$ $(j=1, \ldots, n)$ is given by the exterior forms of degree (p, 0). The Laplace operator Δ can be extended[6] to $\bigwedge^{p,q}(T(D^n))$.

<u>Theorem:</u>

<u>A bihomogeneous form</u> $\alpha \in \bigwedge^{p,q}(T(D^n))$ <u>is harmonic if and only if</u> $q = 0$.

This theorem allows to extend the construction of the Poisson kernel to the non-scalar case. The tensor algebra is defined in the same way as in the case of spin zero (chapter 10)

$$T \bigwedge (T(D^n)) = \bigoplus_m (T(D^n))^{\otimes m}$$

We consider now the spinor fields on D^n: the main feature of the following construction is to define the spinor field on $M^n = P(M^n)/SO(1, n-1)$ (see chapter 1) and to induce a field on D^n by using the stereographic projection $F : T^n \to D^n$ (chapter 2) where $T^n = \mathbb{R}^n + i V^n$, $V^n = \{ y \in \mathbb{R}^n \mid y_1^2 - \ldots - y_n^2 > 0,\ y_1 > 0 \}$.

The condition on a system of n matrices γ_i of order N:

$$\left(\sum_{i=1}^{n} y_i \gamma_i \right)^2 = \left(y_1^2 - \ldots - y_n^2 \right) I_N$$

is equivalent to:

$$\{ \gamma_1 = I_N , \ \gamma_i \gamma_j + \gamma_j \gamma_i = 0 \ (i \neq j), \ \gamma_i^2 = -I_N \ (i = 2, \ldots, n) \ \ \bar{\gamma}_i^T = \gamma_i \ \}$$

(I_N = unit matrix of order N).

For $n = 2p+1$, the values of N are multiples of 2^p and the elements γ_i generate the Clifford algebra $C(T(D^n))$.

We construct now the elementary solution S_n of the Dirac equation and its Fourier transform; S_n satisfies by definition the equation

$$\left(\sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i} \right) S_n = \delta \, I_{2^p}$$

If we introduce the elementary solution E_n of the scalar wave equation (chapter 5) we obtain:

$$S_n = \left(\sum \gamma_i \frac{\partial}{\partial x_i} \right) E_n \, I_{2^p}$$

because

$$\left(\sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i} \right) S_n = \left(\sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i} \right)^2 E_n \, I_{2^p} = \delta \, I_{2^p}$$

The Fourier transform $\hat{S}_n(k)$, given by

$$\hat{S}_n(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int S_n(x) \, e^{i \langle k, x \rangle} \, dx_1 \ldots dx_n$$

satisfies the equation $\left(\sum_{i=1}^{n} \gamma_i k_i \right) \hat{S}_n(k) = I_{2^p}$

Therefore

$$\hat{S}^2(k) = \frac{\sum \gamma_i k_i}{(k_1^2 - \ldots - k_n)^2} = \hat{E}_n(k) \, I_{2^p}$$

The map $F : T^n \to D^n$ of the complex light cone T^n on the bounded domain D^n gives the same relation

$$S_n^2(z,\xi) = P_n(z,\xi)\, I_{2^p}$$

between the elementary solutions $S_n(z,\xi)$ and $P_n(z,\xi)$ of the Dirac- and Laplace operators on D^n. According to the results of chapter 5

$$P_n(z,\xi) = \frac{V(D^n)^{\frac{1}{2}}}{V(Q^n)} \frac{(1+|zz'|^2 - 2\bar{z}z')^{\frac{n}{2}}}{|(z-\xi)(z-\xi)'|^n}$$

and therefore the coefficient of the elementary solution $S_n(z,\xi)$ is $\left(\frac{V(D^n)^{\frac{1}{2}}}{V(Q^n)}\right)^{\frac{1}{2}}$.

Application:[16]

The structure constant α, which measures the elementary charge, is interpreted as coefficient of the Green function of the Dirac equation in momentum space.

The invariance group of the Dirac equation

$$\left(\sum_{i=1}^{5} \gamma_i \frac{\partial}{\partial x_i}\right)\psi = 0$$

in the space of signature (4, 1) (the coordinate x_5 being conjugate[15] to the mass m) is invariant under the conformal group $C(M^5) \cong SO(5,2)$. In the representation space $C\,H(D^5)$ of the spinor forms on $D^5 = SO(5,2)/SO(5)\times SO(2)$, the coefficient of the Fourier transform of the elementary solution is

$$\frac{1}{(2\pi)^{\frac{5}{2}}} \frac{V(D^5)^{\frac{1}{4}}}{V(Q^5)^{\frac{1}{2}}} = \left(\frac{1}{3\sqrt{3}}\right) \frac{9}{8\pi^4} V(D^5)^{\frac{1}{4}}$$

which gives[16] the value of the structure constant α.

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