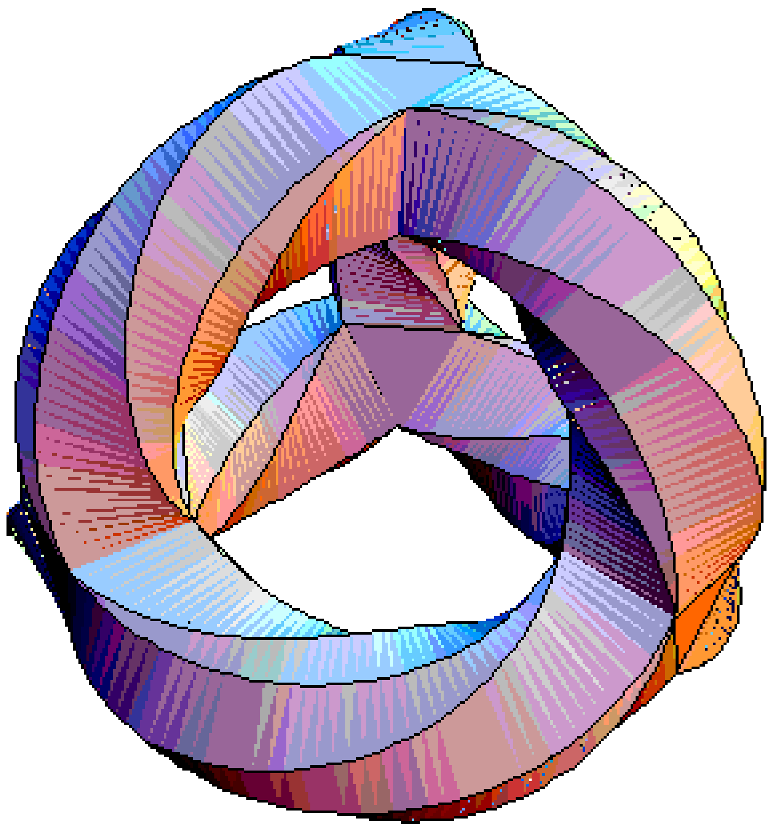
Physics of the Klein Quartic

To make a physics model based on the Klein Quartic, start with 336-element SL(2,7), which double covers the



Klein Quartic (animated image by Greg Egan) . According to Coxeter's book Complex Regular Polytopes (2nd edition, Cambridge 1991), **the 48-element binary octahedral group <4,3,2> is a subgroup of index 7 in SL(2,7)**.

Since the basic building block of the Klein Quartic covering SL(2,7) is <4,3,2>, t**he first task is to see**

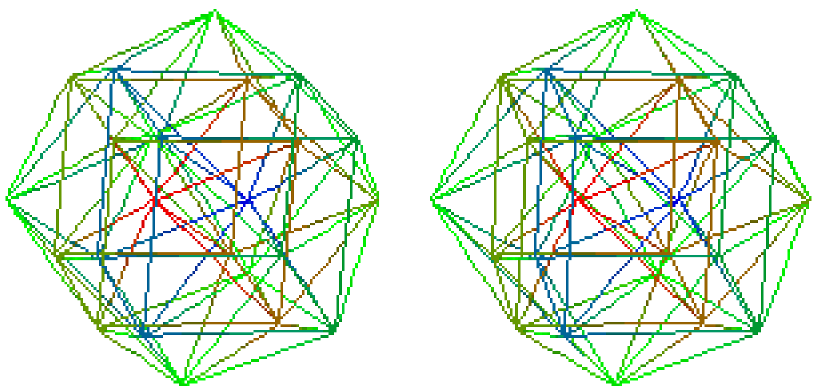
the structure of <4,3,2> and how it might be related to a physics model.

Now consider the 24-element binary tetrahedral group <3,3,2> as a normal subgroup of 48-element <4,3,2> and look at the coset space <4,3,2> / <3,3,2> .

According to Coxeter, given quaternionic space with basis {1,i,j,k} , <3,3,2> = the 24 vertices

- +/- 1, +/- i, +/- j, +/- k, and
- (1/2)(+/- 1 +/- i +/- j +/- k)

of the 24-cell



and

<4,3,2> = the 24 + 24 = 48 vertices

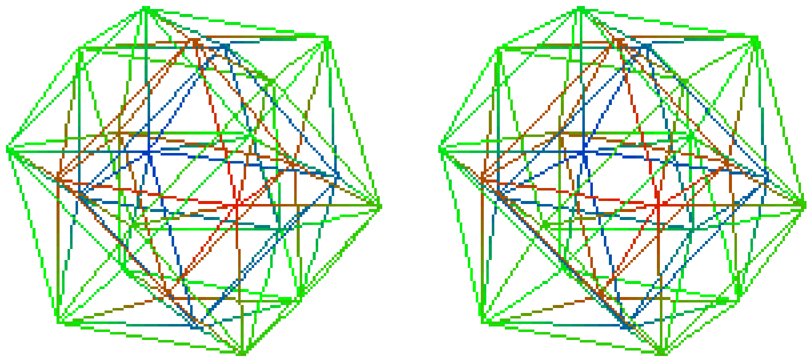
- +/- 1, +/- i, +/- j, +/- k,
- and (1/2)(+/- 1 +/- i +/- j +/- k)

of the 24-cell

plus

- (1/2)(+/- 1 +/- i) and
- (1/2)(+/- 1 +/- j) and
- (1/2)(+/- 1 +/- k) and
- (1/2)(+/- i +/- j) and
- (1/2)(+/- j +/- k) and
- (1/2)(+/- k +/- i)

of the dual/reciprocal 24-cell



Therefore,

the coset space <4,3,2> / <3,3,2> is represented by the vertices

- (1/2)(+/- 1 +/- i) and
- (1/2)(+/- 1 +/- j) and
- (1/2)(+/- 1 +/- k) and
- (1/2)(+/- i +/- j) and
- (1/2)(+/- j +/- k) and
- (1/2)(+/- k +/- i)

of the dual/reciprocal 24-cell.

The next task is to see whether the coset space has a natural group structure, and if so, what it is.

Coxeter says (here I use the notation Cm for the cyclic group of order m): "... every finite reflection group has a subgroup of index 2 which is a rotation group, generated by products of pairs of reflections. ... A convenient symbol for this rotation group of order 2s is (p,q,r) ...

- the tetrahedral group (3,3,2) of order 12,
- the octahedral group (4,3,2) of order 24,
- the icosahedral group (5,3,2) of order 60

... are subgroups of index 2 in ...

- [3,3]
- [4,3]
- [5,3] respectively ..."

Consider the 12-element tetrahedral rotation group (3,3,2) = A4 and add Euclidean reflections to get the 24-element tetrahedral rotation/reflection group [3,3] .

As John Baez says: "... The tetrahedral rotation/reflection group [3,3] is isomorphic to the octahedral rotation group (4,3,2). ...".

So, look at the coset space <4,3,2> / (4,3,2)

Given that 24-element (4,3,2) = [3,3]

and that Coxeter says (where Cm denotes the cyclic group of order m):

"... [3,3] ... of order [24] ... yields

(C4/C2 ; <4,3,2> / <3,3,2>) = GL(2,3) ... of order ...[48]...

... Apart from the little complication caused by the common element -1 of ... C4 ... and ... <4,3,2> ... we have here an instance of a 'subdirect product' (Hall 1959 ... The Theory of Groups ... pp. 63-4) ...", we see that, apart from the complication noted by Coxeter, and another complication due to factoring out the C4/C2 part,

<4,3,2> / <3,3,2> = [3,3] = (4,3,2) = S4 = permutations of 4 elements

so that an intuitive picture (subject to the indicated complications) is that <4,3,2> is made up of <3,3,2> plus (4,3,2) = S4 or in other words <4,3,2> is made up of the 24 vertices

- +/- 1, +/- i, +/- j, +/- k, and
- (1/2)(+/- 1 +/- i +/- j +/- k)

of the 24-cell

plus the S4 permutations of the 4 quaternion basis elements {1,i,j,k}

Here is a physical interpretation:

For the 24-cell,

- +/- 1, +/- i, +/- j, +/- k correspond to 8-dim spacetime
- (1/2)(+ 1 +/- i +/- j +/- k) correspond to 8 fermion particles
- (1/2)(- 1 +/- i +/- j +/- k) correspond to 8 fermion anti-particles

For the dual 24-cell,

24 gauge bosons corresponding to the elements of S4, which, according to Barry Simon's YABOGR book, more formally titled Representations of Finite and Compact Groups AMS Grad. Stud. Math. vol 10 (1996), are:

- e^1
- (12)^6
- (12)(34)^3
- (123)^8
- (1234)^6

You can see this structure from another point of view by recalling that the root vector diagram of F4 has 48 elements, which are, just as above, the 24-cell and the dual 24-cell.

The 4 Cartan subalgebra elements of F4 should be added to the dual 24-cell root vectors to produce a 28-dim Spin(8) gauge group whose generators correspond to:

- e^1
- (12)^6
- (12)(34)^3
- (123)^8
- (1234)^6
- Cartan^4

As Pierre Ramond said in hep-th/0112261 , "... the triality of ... SO(8) ... links its tensor and spinor representations via a Z3 symmetry. The exceptional group F4 is the smallest which realizes this triality explicitly. ...".

Although I do not agree with Ramond's general superstring-type approach to physics, I quote him as an authority figure to allay fears against putting fermions and bosonic structures in the same algebra, such as F4. In my view, exceptional E and F Lie algebras are effectively in some sense superalgebras that have pure Lie algebra structure.

Since an 8-dim spacetime with a Spin(8) gauge group does not look like the world in which we live, let the model so far correspond to high-energy regions, with our world being described by the model after dimensional reduction of 8-dim spacetime into 4-dim spacetime, with the remaining 4-dim corresponding to a CP2 Kaluza-Klein internal symmetry space, as done by Batakis in Class. Quantum Grav. 3 (1986) L99-L105.

Then, here is what happens to the 28 Spin(8) gauge bosons with generators

- e^1
- (12)^6
- (12)(34)^3
- (123)^8
- (1234)^6
- Cartan^4

The 16 generators

- (12)^6
- (1234)^6
- Cartan^4

produce a U(2,2) = U(1)xSU(2,2) = U(1)xSpin(2,4) Lie algebra which gives gravity by the MacDowell-Mansouri mechanism.

The remaining 12 generators

- e^1
- (12)(34)^3
- (123)^8

produce U(1), SU(2), and SU(3) respectively.

Here, based on e-mail conversation with Garrett Lisi in June 2005, is another way to see what happens to the 28 Spin(8) gauge boson generators:

```
Let F8 denote a Spin(8) bivector 2-form over octonionic 8-dim spacetime.
As Garrett Lisi pointed out, a conventional definition of F8 would be
F8 = d A8 + 1/2 A8 A8, where A8, a Cl(8) 1-form, is regarded as the
fundamental field variable. However, in my physics model, I prefer
to think of bivector gauge bosons not as derived from vectors by d,
but, by using triality, to see them as bivector = bihalfspinor
antisymmetric (+half-spinor, -half-spinor) pairs.
Since the triality isomorphism among vectors and half-spinors
is only available fully in 8-dim, it is one of the reasons
that I think that Cl(8) is the uniquely best building block
for a realistic particle physics model.
Equivalence of those two definitions of F8 may imply a relationship
between spinors and the nilpotent covariant derivative.

Let *8 be the 8-dim Hodge star.
```

```
F8 /\ *8F8  is an 8-form over 8-dim spacetime.

Introduce
a preferred quaternionic subspace 4S that will be 4-dim spacetime.

Now, follow F. Reese Harvey's book Spinors and Calibrations:

The spatial part of 4S is defined by an associative 3-form,
which can be defined by an element g of G2 = Aut(octonions).

g also fixes a coassociative 4-form
that defines an internal symmetry space 4I.

The associative form 4S can be written as
w123 - w156 - w426 - w453 - w147 - w257 - w367

The coassociative form 4I can be written as
w4567 - w 4237 - w1537 - w1267 - w2536 - w1436 - w1425

and we have 4S /\ 4I = 7 w1234567

The F8 bivectors are generators of Spin(8) = G2 + ( vector S7 + spinor S7 )

Fixing a g in G2 reduces G2 to SU(3)
and
because it defines an associative 3-dim subspace of the vector S7
it breaks the vector S7 into associative S3 + coassociative S4
so
we reduce Spin(8) to SU(3) + ( (vector S3 + vector S4) + spinor S7 )

Since the vector S3 belongs to 4S and the vector S4 belongs to 4I
we have,
for the internal symmetry gauge groups after dimensional reduction,
SU(3) + vector S4

If SU(3) is to act globally on the 4I internal symmetry space,
the global structure of 4I should morph from S4 to CP2 = SU(3)/U(2),
which then would give a Batakis-type  8-dimensional Kaluza-Klein
model with CP2 extra dimensions and standard model gauge group
generators SU(3)xU(2) = SU(3)xSU(2)xU(1).

The left-over spinor S7 + associative S3 + G2/SU(3) look like:
spinor S7 = spinor S3 + spinor S4
associative S3
set of G2 associative structures left over after picking g and SU(3).

Now look at how the SU(3) and U(2) generators fit inside the 24-cell root
vector diagram of D4 Spin(8).
First, look at 2-dimensional the root vector diagram
(including Cartan elements in the center)

          yb

    xb           zb

         tb tr
```

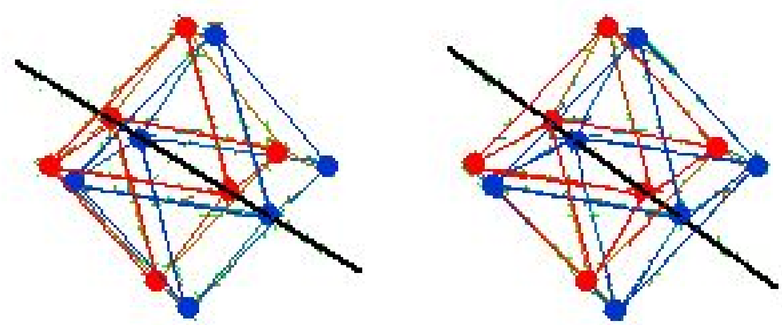
```
    zr           xr

           yr
```

and then blow it up into a 3-dimensional cube

```
  tb----xb
   |\   | \
   | zb----yb
   | |  |  |
  yr-|--zr |
    \|    \|
     xr----tr
```

and then consider the front and back square faces of the cube
as two squares on the base planes, parallel to each other,
of two octahedra

see 2octaimage.jpg

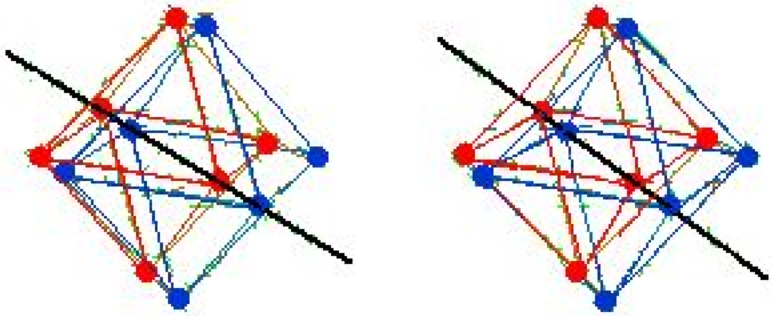


The U(2) root vectors can be represented as four vertices
(including the abelian U(1) and the SU(2) Cartan element)
along a line

```
wr       kr  kb       wb
```

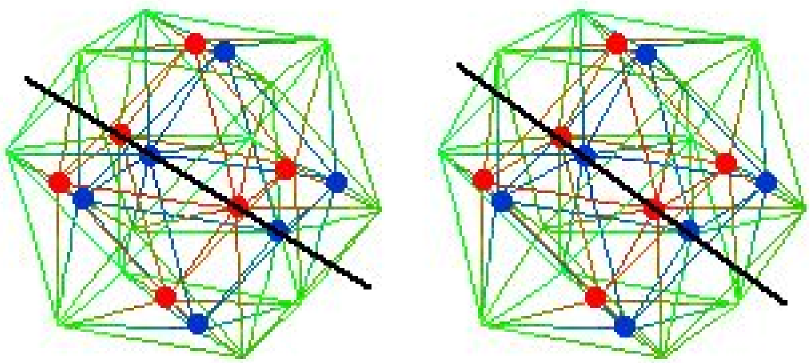
and then consider the line as the common axis perpendicular
to the base planes of the two octahedra, containing the
remaining2+2 = 4 vertices of the two octahedra,

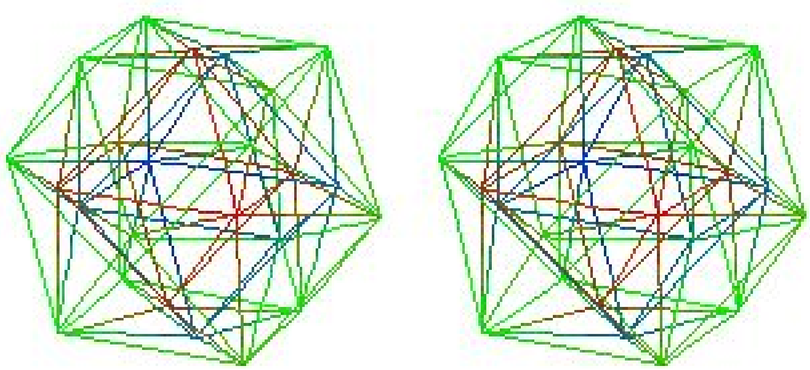
see 2octaimage.jpg



and then consider the two octahedra as part of the 24-cell
root vectors of D4 Spin(8)

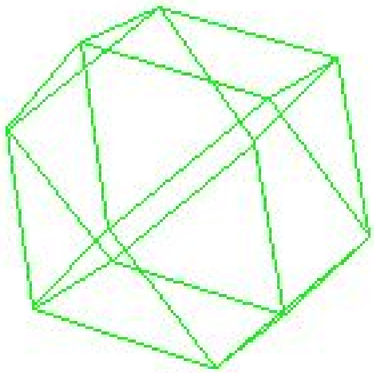
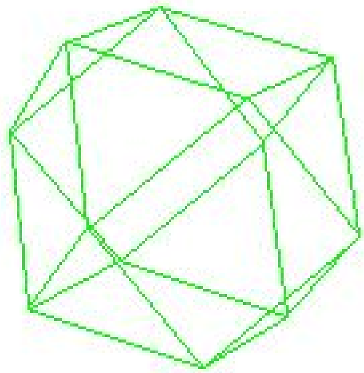
see 2octa24cell.jpg and 24cellD4.jpg





After the Standard Model SU(3)xU(2) vertices are removed
from the D4 Spin(8) 24-cell root vector diagram (including
4 Cartan subalgebra elements at the origin)
what remains is a 3-dim cuboctahedron

see D3cubo.jpg

Since a 3-dim cuboctahdron with 3 origin vertices is
the root vector diagram of D3 = Spin(2,4) = SU(2,2) = A3
and adding a fourth origin vertex gives U(1)xSU(2,2) = U(2,2).

Therefore
the 16 remaining generators give you U(2,2) = U(1) x Spin(2,4)
and Spin(2,4) gives gravity by the MacDowell-Mansouri mechanism.

Those 16 remaining generators correspond to
vector S3
spinor S7 = spinor S3 + spinor S4
the 6 dimensions of G2 associative structures

Then, identify the vector S3 + spinor S3 + spinor S4 with Sp(2)
Anti deSitter Sp(2) = Spin(2,3) has
6 Lorentz and 4 translation-like generators
and (from the point of view of compact signature SU(4))
SU(4) / Sp(2) = 5-dimensional set of quaternionic structures of C4 is
identified with a 5-dimensional subset of the G2 associative structures.
The 6th dimension of the G2 associative structures
gives 16-dimensional U(2,2) with compact version U(4).

To see more explicitly how this all works in terms of the 24-cell
root vector diagram of D4 Spin(8):

The 4 origin root vectors can correspond to

```
01

        23

                45

                        67
```

and the 24-cell vertices can correspond to

```
    02  03  04  05  06  07
    12  13  14  15  16  17
```

```
            24  25  26  27
            34  35  36  37
                    46  47
                    56  57
```

In terms of the basis (x,y,z,t,k,w,r,b), they are

```
xy  xz  xt  xk  xw  xr  xb
    yz  yt  yk  yw  yr  yb
        zt  zk  zw  zr  zb
            tk  tw  tr  tb
                kw  kr  kb
                    wr  wb
                        rb
```

The 6 that involve only (0,1,2,3) = (x,y,z,t) form the Lorentz boosts
and rotations of physical spacetime:

```
xy  xz  xt
    yz  yt
        zt
```

Note that (01,02,12) = (xy,xz,yz) are rotations
and (03,13,23) = (zt,yt,zt) are Lorentz boosts.

Now let them interact with 4 = k.
Moving in 5-space (k-space) can be regarded as
just an "extra" dimension added to 4-dim spacetime
in which "rotations" look like translations,
and we then get the 10 with indices 0,1,2,3,4
which are taken to be generators of deSitter/Poincare gravity

```
01  02  03  04
    12  13  14
        23  24
            34
```

or

```
xy  xz  xt  xk
    yz  yt  yk
        zt  zk
            tk
```

where (04,14,24,34) = (xk, yk, zk, tk) are translations.

Next let the 10 interact with 5 and let 5 have SU(3) color (r+g+b).
Moving in 6-space (w-space) can also be regarded as an "extra" dimension.
In it "rotations" look like conformal transformations
(1 dilation and 4 special conformal transformationsi),
and we then get the 15 with indices 0,1,2,3,4,5
which are taken to be generators of Segal's conformal Spin(2,4) = SU(2,2)
which can also be gauged to produce gravity:

```
01  02  03  04  05
    12  13  14  15
        23  24  25
            34  35
                45
```

or

```
xy  xz  xt  xk  xw
    yz  yt  yk  yw
        zt  zk  zw
            tk  tw
                kw
```

Note that the 45, which is only in (k,w) space,
corresponds to the dilation
and
(05,15, 25, 35) = (xw, yw, zw, tw) are
the 4 special conformal transformations.

That leaves 13 generators left over, those with at least one index 6 or 7

```
06  07
16  17
26  27
36  37
46  47
56  57
    67
```

or

```
xr  xb
yr  yb
zr  zb
tr  tb
kr  kb
wr  wb
    rb
```

Let 67 = rb represent the U(1) phase of particle propagators,
which is the U(1) of 16-dimensional U(1)xSU(2,2) = U(2,2).

That leaves 12 generators:

```
06  07
16  17
26  27
36  37
46  47
```

```
56  57

or

xr  xb
yr  yb
zr  zb
tr  tb
kr  kb
wr  wb
```

and they correspond to the 6+6 = 12 vertices of the two octahedra
described above as giving the standard model SU(3)xU(2), where
SU(3) is represented by

```
          yb

    xb           zb

         tb tr

    zr           xr

           yr
```

(Note that the 3 colors have structure similar
to that of the 3 spatial dimensions.)

and U(2) is represented by

```
wr       kr  kb       wb
```

Note that
the local U(2) action on the 4I internal symmetry space is
totally confined to the kwrb 4I space itself,
while
the global SU(3) action on 4I involves not only the rb color
space of 4I but also the txyz 4S physical spacetime,
somewhat similarly to the actions of the Batakis CP2 Kaluza-Klein model.

How does the Hodge star work in 8-dim and 4-dim spacetimes?

The Hodge star for Spin(8) is defined in 8-dim spacetime
by letting mn be lower indices and MN be upper indices for F
so that *Fmn = (1/2) e(mnabwxyz) FABWXYZ
which is natural because of the Clifford algebra
structure of Cl(8) with Spin(8) generators being the bivectors.

However, it is not so natural in 4-dim spacetime
because the standard model group SU(3)xSU(2)xU(1) is bigger
than the bivector algebra Spin(1,3) of the Clifford algebra Cl(1,3)
of 4-dim Minkowski spacetime,
and

the question arises as to what the 8-dim Hodge star morphs into
when spacetime is reduced to 4-dim.

By looking at the conformal Spin(2,4) that gives gravity
after reduction you can see that its natural Clifford structure
is on a 6-dim vector space so that it would want to have
a 6-dim Cl(2,4) Hodge star, since a 6-dim spacetime is the
spacetime of linear actions of the Clifford-algebra conformal group.

However,
due to the special automorphism Spin(2,4) = SU(2,2),
the conformal group can be written in a way that acts
naturally on a 4-dim spacetime as a unitary group.
Since the standard model groups are also unitary,
all the relevant gauge groups after reduction are unitary:
U(2,2), SU(3), SU(2), and U(1).

Therefore, after reduction, the Hodge star should be the usual one
for conventional Yang-Mills physics theories,
based on the graded exterior algebra structure of SU(n) Lie algebras.
Since that structure is, for example for U(4):

1   4   6   4   1 graded structure
(coinciding with the graded structure of the Clifford algebra
for Cl(1,3) Minkowski spacetime)

where the second 4 can be regarded in terms of the first 4 as *4:
U(4) = 4 (x) *4 is 4x4 = 16-dim
so there is a natural Hodge star for U(4)
and it can be used for subgroups including SU(3), SU(2), and U(1),
and therefore to describe accurately gravity (MacDowell-Mansouri)
and the standard model.

In April 2005 sci.physics.research discussions with John Baez,
Garrett Lisi describes the MacDowell-Mansouri Mechanism this way:

> "... I only have one reason for justifying the use of Clifford
> algebra for this stuff -- it comes unavoidably from the application
> of Occam's razor. ... Describing fermions requires the use of
> a Clifford bundle. So, I figure, since there is no getting out
> of having this Clifford fiber bundle thing,
> might as well make the best possible use of it.
> And what I'm finding is that, along with the vectors and forms
> that almost come for free with a manifold,
> that's ALL one needs to describe the physical fields. ...
> The fermions come in as a Grassman valued section of the Clifford
> bundle as a result of applying the BRST gauge fixing method ...
> ... Let me do the whole thing from the ground up ... :
> The only dynamic variable is the connection:
> A = e + W
> With e the Clifford vector valued 1-form (the frame),
> and W the Clifford bivector valued 1-form (the spin connection).
> The curvature is F = d A + (1/2) A A = Fo + Fe
> where the Clifford odd part of this curvature,
> a Clifford vector valued 2-form, is the torsion
> Fo = d e + W x e

where for two one forms the cross product
is W x e = (1/2)(W e + e W),
but in general for forms of orders that commute
it's A x B = (1/2) (A B - B A).
The even part of this curvature is
the Clifford bivector valued 2-form
Fe = d W + (1/2) W W + (1/2) e e = R + (1/2) e e
The Bianchi identity is d R + W x R = 0.
The action we start with is
S = (1/2) int < F F g > = (1/2) int < Fe Fe g >
which works since g is the Clifford unit 4-vector,
and only 4-vectors times g give a trace.
Plugging in F, this action is equivalent to
S = (1/2) int < R e e g + (1/4) e e e e g >
the GR action, since the other term we get is a boundary term,
< R R g > = < d (W d W + (1/3) W W W) g >
The ONE equation of motion then,
arising from varying A in the action, is
0 = D (Fe g) = d Fe g + A x (Fe g)
  = d R g + (1/2) d e e g +
    + (1/2) (e + W) (R + (1/2) e e) g - (1/2) (R + (1/2) e e) g (e + W)
  = (1/2)((e R + R e) + e e e) + (1/2) (d e e + W x (e e)) g
The third line came from the second using the Bianchi identity.
The first term on the last line, a Clifford odd 3-form,
is Einstein's equation, and the second term on the last line,
a Clifford even 3-form,
is the equation for the torsionless spin connection.
That's pretty dense to read, but it's the whole derivation.
One other thing I've noticed is that
the Hamiltonian formulation of this stuff is pretty nice.
If one is a truly cretinous physicist one can define
the momentum 2-form as
p = (delta / delta A) S = (1/2) (F g + g F) = Fe g
And writing the action as
S = int <p d A - H>
with Hamiltonian 4-form
H = - (1/2) p A A - (1/2) p p g
gives the BF action
S = int < p (F + (1/2) p) g >
with the p equal to the even part of the usual B,
and the sign changed.
To further annoy mathematicians one could write
the equations of motion as
d A = (d/d p) H
d p = (d/d A) H
and cook up a Poisson bracket formulation.
... diffeomorphisms and local frame rotations enter through
the same infinitesimal gauge transformation of the connection:
A' = A + d C + A x C
The weird thing I just got is that the conserved
Noether current 3-form corresponding to this symmetry is
< (d B + A x B) C >
which vanishes because that's the equation of motion.
Does that make sense and mean anything? It seems strange
for a conserved current corresponding to a symmetry to vanish.
It looks like if I include the whole Clifford bundle connection

we get Einstein-Cartan theory (non-vanishing torsion)
and a couple of other gauge fields. ...
... one cool thing we could do in the complex Cl ... that's
build the duality projector:
P = (1/2) (1 - i g)
So we could then work in the "sef-dual" formulation of GR
by just using the self-dual half of the action,
S = i < Fe P Fe >
And a lot of people like to do that,
Plebanski and Ashtekar for example. ...".

A question that arises is how does the
1   8 28 56 70 56 28   8   1 Clifford Hodge star
morph into the
1   4   6 *4 *1   unitary/exterior Hodge star ?
(note that the 6 can be written as 3 + *3)

The way I see it is that the 8-dim Hodge star uses
an 8-dim txyzabcd pseudoscalar,
the first 4 terms txyz of which will go to physical spacetime
and the second 4 terms abcd of which will go to CP2 Kaluza-Klein space,
and
that the first 4 terms txyz will go to the first unitary/exterior 4
and the last 4 terms abcd will go to the dual unitary exterior *4.

A closely related questin is
how is the low-energy g_munu curvature "embedded"
in the flat-looking Spin(8) up in Cl(8) 8-dim high-energy spacetime ?

In 8-dim at high energies my Cl(8) model is flat with
no dynamic relativity-type g_munu curvature,
and so no nontrivial g_munu raising and lowering.

Dynamic g_munu curvature only appears after:
1 - picking a quaternionic subspace splits 8-dim spacetime
into 4-dim Minkowski physical spacetime and 4-dim internal
symmetry space that is CP2;
2 - the U(2,2) part of Spin(8) acts by MacDowell-Mansouri
to produce Einstein etc gravity, by which process
the dynamic g_munu curvature emerges and you can then
do raising and lowering with the dynamic g_munu.

In the above-mentioned April 2005 sci.physics.research conversation
of **Garrett Lisi** with John Baez, John Baez said

"... If you're working over the complex numbers,
as evil physicists usually do,
the Clifford algebra Cl_4 is isomorphic to the algebra
of 4x4 complex matrices.
So, if we think of it as a Lie algebra via [a,b] = ab-ba,
we get the Lie algebra of 4x4 complex matrices,
usually known as gl(4,C).
But this is a complexification of u(4),
and since evil physicists never even *care* about
the difference between real and complex Lie algebra,

I can easily imagine someone saying that the answer is u(4). ..",

and **Garrett Lisi** said

"... I'm more than happy to work with real Clifford algebras. ...".

Since real Clifford algebras have the periodicity 8 property
that I need for my physics model for such things as building
the generalized hyperfinite II1 von Neumann algebra factor
(In this I may be sloppy about signature.):

At Cl(8) high energy, we have 28 Spin(8) as gauge group.

Spin(8) has a natural U(4) subgroup (for MacDowell-Mansouri gravity)
and
the standard model groups correspond to
the coset space Spin(8) / U(4) which is the set of complex structures
on R8 that are compatible with its Euclidean structure (see
Besse, Einstein Manifolds (Springer 1987)).

As John Baez said,
consider U(4) as roughly M(4,C) the 4x4 complex matrices.

M(4,C) is the real Clifford algebra Cl(2,3) of the anti-deSitter group
Spin(2,3) = Sp(2) that is the basis of the MacDowell-Mansouri mechanism,
which uses the bivector 10 of the Cl(2,3) grading 1 5 10 10 5 1
to get gravity (4 of that 10 giving gravity and 6 for torsion).

So,
the dynamic g_munu of 4-dim gravity comes from 4 of the bivector 10 of U(4)
which U(4) in turn is embedded in the 8-dim Spin(8).

As to how the U(4) fits inside the Spin(8),
look at the e0, e1, e2, e3, e4, e5, e6, e7 of Cl(8)
and see the U(4) as generated by
e0 - i e1
e2 - i e3
e4 - i e5
e6 - i e7
Let the Spin(8) Dirac operator is d8 + *d8 (where * is Cl(8) pseudoscalar)
Then, with the map #: i -> -i, d8 for Spin(8) goes to d4 + #d4 for U(4).

The U(4) subgroup of Spin(8) has a natural Sp(2) subgroup
(for MacDowell-Mansouri gravity)
and
the 5 special conformal generators related to Higgs
correspond to the 5-dim coset space SU(4) / Sp(2) which
is the set of quaternionic structures on C4
that are compatible with its Hermitian structure (see
Besse, Einstein Manifolds (Springer 1987)).

The 6th element of U(4) containing Sp(2)
is the U(1) of U(4) = U(1) x SU(4),
which in my model corresponds to
the complex phase of particle propagators.

```
Taking that into account, there is a nesting of coset spaces
Spin(8) / U(4) =  complex structures = standard model generators
SU(4) / Sp(2) = quaternionic structures = conformal Higgs stuff
Sp(2) = MacDowell-Mansouri gravity (with torsion if you use all 10 dim).

As to how the Sp(2) fits inside the Spin(8),
look at the e0, e1, e2, e3, e4, e5, e6, e7 of Cl(8)
and see the Sp(2) as generated by
e0 - i e1 - j e2 - k e3
e4 - i e5 - j e6 - k e7
where k = ij (quaternion imaginaries
Let the Spin(8) Dirac operator is d8 + *d8 (where * is Cl(8) pseudoscalar)
Then, with the maps #: i -> -i and $: j -> -j
d8 for Spin(8) goes to d2 + #d2 + $d2 + #$d2 for Sp(2).

Some further interesting questions are:

Is there a duality between Minkowski spacetime and K-K CP2 ?

Can the *4 (or K-K CP2) part be thought of as an imaginary part of
a complex space of which Minkowski spacetime is a real part ?

Are there shadows of D4 triality that can be seen in conformal D3=A3
in the details of the relationship between the Clifford 6-dim D3
and the unitary/exterior 4-dim A3
and
can those remnants of triality be used to establish relations
(after dimensional reduction)
among spinor fermions, gauge bosons, and torsion?
```

Therefore,

Klein Quartic Physics gives a 4-dim spacetime with CP2 Kaluza-Klein and gravity and the U(1)xSU(2)xSU(3) Standard Model.

As to the fermions,

- the (1/2)(+ 1 +/- i +/- j +/- k) 8 fermion particles and
- the (1/2)(- 1 +/- i +/- j +/- k) 8 fermion anti-particles

give the Standard Model first generation, with the second and third generations being given by considering how the fermions move in the 4-dim spacetime and/or the Kaluza-Klein CP2.

See Sets2Quarks9.html#sub13 for details of how that works.

Why does SL(2,7) need 7 copies of <4,3,2> ?

Having gotten F4 structure out of Klein Quartic Physics, consider further structures:

F4 can be complexified to get E6, and the E6 5-grading

g = E6 = g(-2) + g(-1) + g(0) + g(1) + g(2)

such that

- g(0) = so(8) + R + R
- dimR g(-1) = dimR g(1) = 16 = 8 + 8
- dimR g(-2) = dimR g(2) = 8

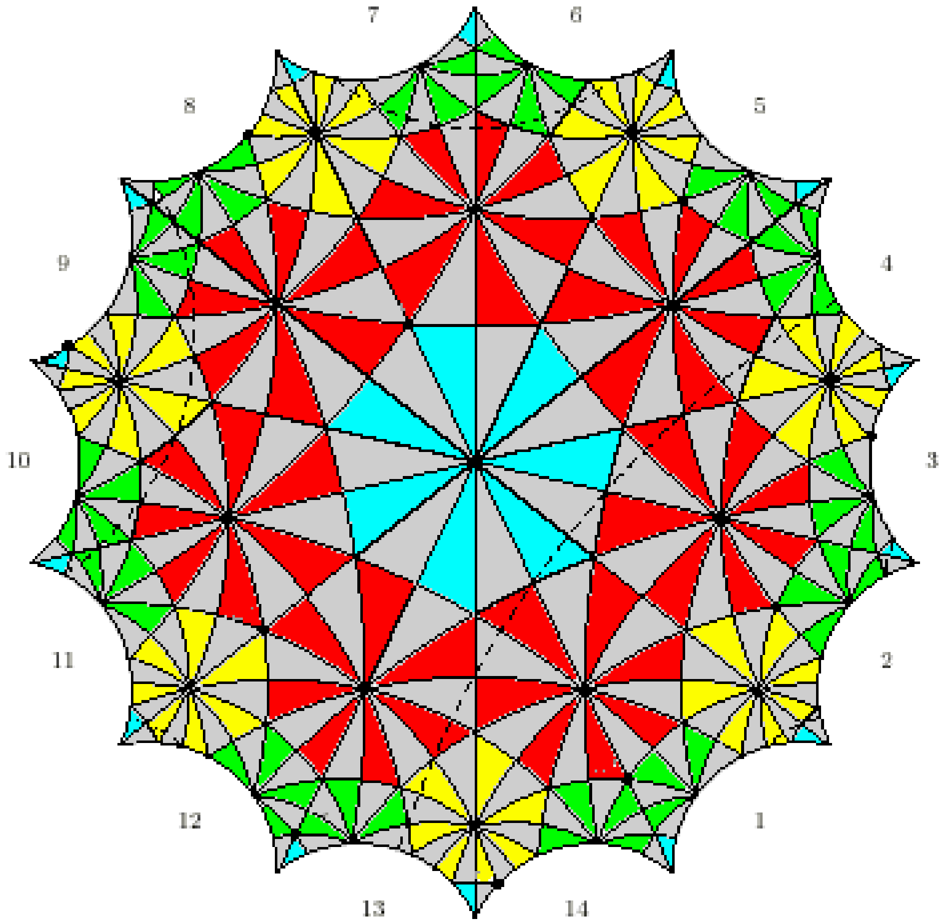
is useful in seeing that the fermionic part of the model lies in the odd part of the E6 5-grading, and is useful in interpreting the model with respect to string theory, as described in CERN-CDS-EXT-2004-031 which is at http://cdsweb.cern.ch/ .

In that E6 string model, each Planck-scale E8 lattice D8 brane is a superposition / intersection / coincidence of eight E8 lattices.

7 of the 8 lattices are independent E8 lattices, each corresponding to one of the 7 imaginary octionion basis elements i, j, k, E, I, J, K.

The 8th E8 lattice is dependent on the 7, and can be thought of as corresponding to the real octonion basis element 1.

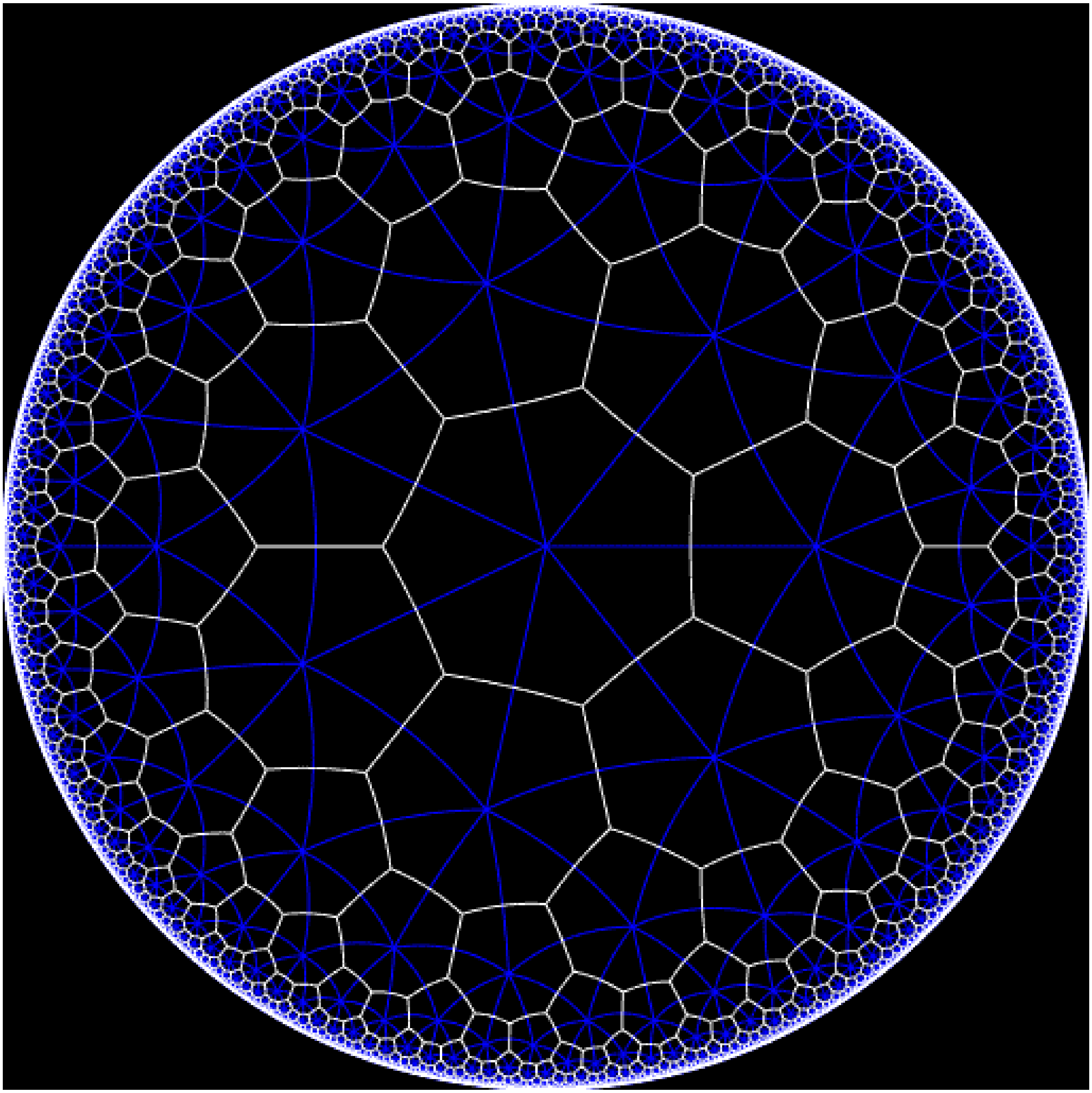
The representations of the Klein Quartic by triangles



may be related to the fact that the 7 imaginary octonions, and therefore the 7 coset spaces of SL(2,7) / <4,3,2> , correspond to the 7 associative 3-dimensional quaternionic triangles:

```
                     i
                    / \
                   k---j

  i     j     k     I     J     K
 / \   / \   / \   / \   / \   / \
K---J I---K J---I E---i j---E k---E
```

The representations of the Klein Quartic by heptagons (image from Don Hatch)



may be related to the fact that the 7 imaginary octonions, and therefore the 7 coset spaces of SL(2,7) / <4,3,2> , also correspond to the 7 octonionic heptavertons / Onarhedra. Arthur Young, in his book The Reflexive Universe (Robert Briggs Associates 1978), says: "... The Heptaverton: Connecting seven points each to each requires 21 lines or edges. ... This figure can be thought of as adding a point at the center of the Octahedron, and this additional point creates a set of 6 compressed diagonals in addition to the 15 ...[12 edges plus 3 full diagonals of the octahedron]...".

The outer hull of a heptaverton / Onarhedron is an octahedron.

It has 4 pyramids (half of the 8 pyramids of a simplicial decomposition of the octahedron) plus 4 triangles that make up the other faces of the outer octahedron.

All 4 pyramids share 1 central vertex and the 6 outer vertices are each shared by 2 pyramids and 2 triangles.

The term Onarhedron comes from a later rediscovery, coming from studying octonions, of the heptahedron by Onar Aam and his namesake, Onar, the high god in Norse mythology who created the universe. Here are the 7 heptavertons / Onarhedra:

```
                                     E
                                    / \
                                   /   \
                                  J-----j             E j
         J        I---j         / |     | \           |/
I  -->  / \  -->  |   | -->   i   |  I  |   i   =  J--I--k
       i---K      k---E         \ |     | /          /|
                                  K-----k           K i
                                   \   /
                                    \ /
                                     E


                                     E
                                    / \
                                   /   \
                                  K-----k             E k
         j        J---i         / |     | \           |/
J  -->  / \  -->  |   | -->   j   |  J  |   j   =  K--J--i
       I---K      k---E         \ |     | /          /|
                                  I-----i           I i
                                   \   /
                                    \ /
                                     E


                                     E
                                    / \
                                   /   \
                                  I-----i             E i
         J        K---i         / |     | \           |/
K  -->  / \  -->  |   | -->   k   |  K  |   k   =  I--K--j
       I---k      j---E         \ |     | /          /|
                                  J-----j           J k
                                   \   /
                                    \ /
                                     E


                                     k
                                    / \
                                   /   \
```

```
                                  I-----J             k J
         I        J---j         / |     | \           |/
i  -->  / \  -->  |   | -->   j   |  i  |   j   =  I--i--E
       E---i      K---k         \ |     | /          /|
                                  K-----E           K j
                                   \   /
                                    \ /
                                     k
```

```
                                     k
                                    / \
                                   /   \
                                  J-----I             k I
         J        K---k         / |     | \           |/
j  -->  / \  -->  |   | -->   i   |  j  |   i   =  J--j--E
       E---j      I---i         \ |     | /          /|
                                  K-----E           K i
                                   \   /
                                    \ /
                                     k
```

```
                                     i
                                    / \
                                   /   \
                                  K-----J             i J
         K        I---i         / |     | \           |/
k  -->  / \  -->  |   | -->   j   |  k  |   j   =  K--k--E
       E---k      J---j         \ |     | /          /|
                                  I-----E           I j
                                   \   /
                                    \ /
                                     i
```

```
                                     I
                                    / \
                                   /   \
                                  J-----k             I k
         j        I---J         / |     | \           |/
E  -->  / \  -->  |   | -->   i   |  E  |   i   =  J--E--j
       i---k      K---E         \ |     | /          /|
                                  K-----j           K i
                                   \   /
                                    \ /
                                     I
```

Heptavertons / Onarhedra may be useful in constructing in 4-dimensional spacetime with 3 space dimensions a quantum cellular automata model, or generalized Feynman Checkerboard model, because, just as

- 8-dimensional E8 lattices can be formed by Witting polytopes, and
- 4-dimensional space can be filled by 24-cells, and
- 2-dimensional space can be filled by hexagons, triangles, or squares,

so 3-dimensional space can be filled by octahedra and cuboctahedra, and Onar Aam has shown that onarhedra and cuboctahedra consistenly fit together to form an onarhedral lattice that tiles 3-dim space.

```
K---k---E---I---i---K---J---j---E---i---I---k---J---E---j---K---I
|       |   +   |       |   +   |       |   +   |       |   +   |
j       j+++i+++k       k+++E+++K       K+++J+++i       i+++I+++E
|       |   +   |       |   +   |       |   +   |       |   +   |
I---k---J---E---j---K---I---J---k---i---j---I---K---E---k---j---i
|   +   |       |   +   |       |   +   |       |   +   |       |
K+++J+++i       i+++I+++E       E+++j+++J       J+++k+++I       I
|   +   |       |   +   |       |   +   |       |   +   |       |
j---I---K---E---k---j---i---J---K---k---E---I---i---K---J---j---E
|       |   +   |       |   +   |       |   +   |       |   +   |
J       J+++k+++I       I+++K+++j       j+++i+++k       k+++E+++K
|       |   +   |       |   +   |       |   +   |       |   +   |
E---I---i---K---J---j---E---i---I---k---J---E---j---K---I---J---k
|   +   |       |   +   |       |   +   |       |   +   |       |
j+++i+++k       k+++E+++K       K+++J+++i       i+++I+++E       E
|   +   |       |   +   |       |   +   |       |   +   |       |
J---E---j---K---I---J---k---i---j---I---K---E---k---j---i---J---K
```

This is a chessboard lattice of onarhedra and cuboctahedra. The 2D cross made of +'es are the onarhedra and the "empty" squares next to them are "flattened" cuboctahedra. If you deform the above lattice by contracting the cuboctahedra so that the similar imaginaries on the opposite sides merge, you get two interwovern onarhedral lattices.

You can also use onarhedra to get triangular tilings that may be similar to tilings of tetrahedra and truncated tetrehedra:

```
i___E___J___I___j___k___K___i
 \*/ \*/ \*/ \*/ \*/ \*/ \*/ \
  I___j___k___K___i___E___J___I
 / \*/ \*/ \*/ \*/ \*/ \*/ \*/
k___K___i___E___J___I___j___k
 \*/ \*/ \*/ \*/ \*/ \*/ \*/ \
  E___J___I___j___k___K___i___E
 / \*/ \*/ \*/ \*/ \*/ \*/ \*/
I___j___k___K___i___E___J___I
 \*/ \*/ \*/ \*/ \*/ \*/ \*/ \
  K___i___E___J___I___j___k___K
```

The * is inside the octonionic triangles, those pointing downward. The triangles pointing upward are not associative, but they can be made into co-associative tetrahedra by adding an appropriate imaginary above them. This tiling is self-similar in that, if you replace each triangle with its corresponding imaginary, then you get exactly the same tile.

The cuboctahedron-onarhedron tiling uses the interior coassociative squares of Onarhedra, while the triangular tiling uses the exterior associative triangles.

These tilings might be useful to construct something like a spin network or a spin foam model or a discrete spacetime model. Roger Penrose, in the USA edition (Knopf 2005) of his book The Road to Reality, says:

> "... The original ... spin-network proposal ... was ... of a completely discrete character, but the standard loop-variable picture is still dependent upon the continuous nature of the 3-surface in which the 'spin networks' are taken to be embedded. ... a spin foam ... can [be] picture[d] ... as a time-evolving spin network. ... Other suggestions take spacetime to have a discrete periodic lattice structure ... schemes like Raphael Sorkin's causal-set geometry ... [take]... spacetime ... to consist of a discrete, possibly finite, set of points for which the notion of causal connection between points is taken to be the basic notion. ... Other ideas ... arise from ... quaternionic geometry ... octonionic ... physics ... etc. ...".

With respect to spin foam models and how E6 and F4 might be used in them, John Baez said (in some spr posts):

> "... Taking the quotient (structure group) / (automorphism group) we get homogeneous spaces of the sort used to construct spin foam models of quantum gravity. ... with a certain real form of E6 and the compact real form of F4 ...[in the case of H3(O), the quotient (structure group) / (automorphism group) = E6 / F4]... the quotient of Lie groups E6 / F4 is what matters for the spin foam models, and this is a bit "curvier" ...[than the]... quotient of Lie algebras e6 / f4 [which] is a vector space that can be naturally identified with H3_0 (O) [the traceless subalgebra of the 27-dim octonionic Jordan algebra H3(O)]... e6 / f4 can be viewed as a tangent space of E6 / F4. ...".

Note that F4 is the automorphisms of the exceptional Jordan algebr J3(O) and E6 is the automorphisms of the Freudenthal algebra Fr(3,O).

I am happy that Klein Quartic Physics as I have described it seems to me to

be not only consistent with, but also equivalent to, my D4D5E6E7E8 VoDou Physics model.

Quaternionic SL(2,3) and Octonionic SL(2,7)

Since

- 1 copy of binary tetrahedral of tetrahedron = 1x24 = 24-element SL(2,3) and
- 7 copies of binary octahedral of cube = 7x48 = 336-element SL(2,7)

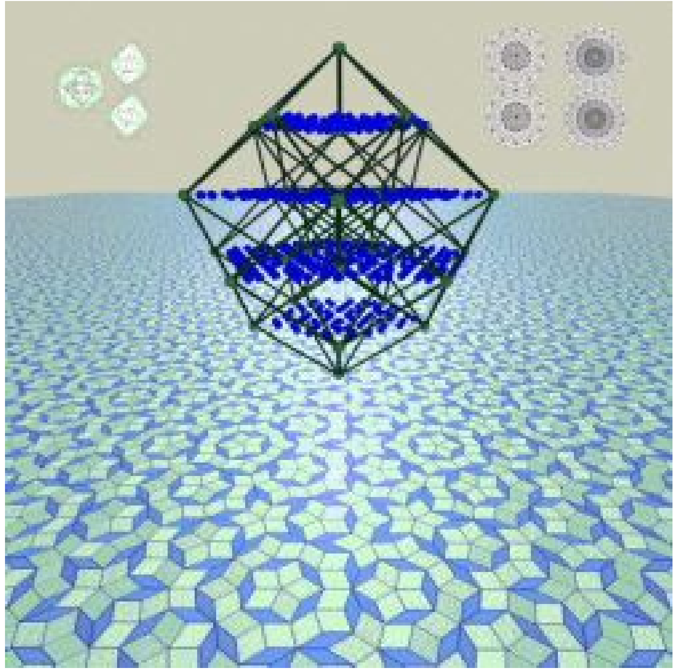
then it seems to me that it is probably true that

- the 4 faces of the tetrahderon are like the 4 dimensions of quaternions with the 3 of SL(2,3) being the 3 quaternion imaginaries and the 1 copy is due to the 1 associative triangle that can be constructed from the 3 imaginary quaternions, and
- the 8 faces of the octahedron are like the 8 dimensions of octonions with the 7 of SL(2,7) being the 7 octonion imaginaries and the 7 copies are due to the 7 associative triangles that can be constructed from the 7 imaginary octonions.

Klein Quartic Physics and the McKay Correspondence

The McKay correspondences (using his notation of <r,q,p> instead of <p,q,r>) include:

```
D4 corrresponds to <2,2,2> = quaternion group of order 4+4 = 8
E6 corresponds to <2,3,3> = 2.Alt[4] binary tetrahedral group of order 12+12 = 24
E7 corresponds to <2,3,4> = 2.Symm[4] binary octahedral group of order 24+24 = 48
E8 corresponds to <2,3,5> = 2.Alt[5] = SL(2,5) binary icosahedral group of order 60+60 = 
120
By root vector structure, in the sense that F4 = D4 + 8 + 16 = D4 + <2,3,3> ,
```



```
D5 = D4 + 1 + (8+8) = D4 + 1 + 2.<2,2,2>

E6 = D5 + 1 + (16+16) = D5 + 1 + (4+4) + (12+12) = D5 + 1 + <2,2,2> + <2,3,3> =
= D4 + 1 + (8+8) + 1 + (4+4) + (12+12) = D4 + 2 + (8+8+8) + (12+12) =
= D4 + 2 + (24+24) = D4 + 2 + <2,3,4> =
= D4 + 1 + (8+8) + 1 + (16+16) = D4 + 8 + 16 + 2 + (8+16) = F4 + 2 + <2,3,3>

E7 = E6 + 1 + (27+27) = E6 + 1 + (3+3) + (24+24) = E6 + 1 + (3+3) + <2,3,4> =
= E6 + 3 + (1+3 + 24+24) = E6 + 3 + F4

E8 = E7 + 1 + (56+1+56+1) = E7 + 3 + (56+56) = E7 + 3 + (30+30) + (24+2+24+2) =
= E7 + 7 + (2,3,5) + <2,3,4> =
= E7 + 3 + Alt[5] + (4+24+24) = E7 + 3 + PSL(2,5) + F4

Note that (2,3,5) = Alt[5] = PSL(2,5) is a simple group as
it is an alternating group of at least 5 elements.
```

Visualization of the Klein Quartic

According The Eightfold Way: The Beauty of Klein's Quartic Curve, edited by Silvio Levy (MSRI Publications -- Volume 35, Cambridge University Press, Cambridge, 1999):

"... The Klein surface is the Riemann surface of the algebraic curve with equation ... x^3 y + y^3 z + z^3 x = 0 ... that ... is mapped into itself by 168 analytic transformations. Since the equation is real, the surface is also mapped on itself by complex conjugation, which can be composed with the analytic maps to give a further 168 antianalytic mappings, yielding a group of order 336. Klein concentrated ... on the subgroup of index 2 and order 168 ... [The Klein Quartic group PSL(2,7) = PSL(3,2) of order 168]... is the second smallest simple noncommutative group. ...

> [According to The Classification of the Finite Simple Groups, by Gorenstein, Lyons, and Solomon (AMS Surveys and Monographs Vol. 40, No. 1, 1994), the smallest is PSL(2,5) = A5 = of order 60 , and some others are PSL(2,8) of order 504 and PSL(2,10) = A6 of order 360.]

... Klein ... approached the ... group and Riemann surface ... by studying the modular group GAMMA(1) of all functions z -> (p z + q) / (r z + s) where p,q,r,s are in Z, and ps - qr = 1. These are permutations of the upper half-plane U ... The upper half-plane is a Riemann surface, so its quotient surface U / GAMMA(1) is also a Riemann surface - a sphere with one ... puncture. ... The congruence subgroups GAMMA(n), which consist of [such] mappings ... such that

```
p   q
        = +/- Id (mod n)
r   s
```

are ... kernel[s] of a homomorphism, GAMMA(n) is a normal subgroup of GAMMA(1), and the factor group acts on the quotient surface as a group of automorphisms. The quotient surfaces for GAMMA(2), GAMMA(3), GAMMA(4), and GAMMA(5), are spheres with 3, 4, 6, and 12 punctures. The factor groups include the symmetry groups of the platonic solids (tetrahedron, octahedron and icosahedron). The quotient surface of GAMMA(6) ...[is]... a torus with twelve punctures ... the factor group GAMMA (1) / GAMMA(6) is rather dull. ... At GAMMA(7) ... Klein found ...[a]... surface ...[of]... genus 3 with 24 punctures. The punctures are "removabele singularities" ... so he had a Riemann surface of genus 3 with 168 automorphisms. The quotient group is ... PSL(2,7) ... The Riemann surface of ...[the Klein Quartic]... is a 168-sheeted covering of the sphere, branched over three points of the sphere.

- Above one of these points the 168 sheets join together in sevens to geve 24 points of the surface. These are the points of inflection. They are also the Weirstrass points.
- Above another branch point, there are 84 points of the surface, where the sheets join in twos. These are the sextactic points, through which pass a conic section that has six-fold contact with the curve.
- Above the third branch point the sheets join in threes to give 56 points of the surface. These 56 points are the of contact of ...[the Klein Quartic]... with the 28 bitangents, or lines that are tangent to the curve at two points.

... The numbers 2, 3, 7 reflect the fact that the universal cover of the whole picture is the triangle group (2,3,7) acting on ... the upper half-plane ... U. The modular group GAMMA(1) is the triangle group (2,3, oo). Replacing oo by 7 amounts to removing the removable singularities. ...

... Fricke discovered the ... group PSL(2,2^3) of order 504 and genus 7 ...".

Roger Penrose's book The Road to Reality comes in two editions:

- UK edition (ISBN: 0224044478, Publisher: Jonathan Cape, July 29, 2004) and
- USA edition (ISBN: 0679454438, Publisher: Knopf, February 22, 2005).

T**he two editions are NOT identical**. For example:

The UK edition on page 1050 says in part: "... Bibliography ... There is one major breakthrough in 20th century physics that I have yet to touch upon, but which is nevertheless among the most important of all! This is the introduction of arXiv.org, an online repository where physicists ... can publish preprints (or 'e-prints') of their work before (or even instead of!) submitting it to journals. ...as a consequence the pace of research activity has accelerated to unheard of heights. ... In fact, Paul Ginsparg, who developed arXiv.org, recently won a MacArthur 'genius' fellowship for his innovation. ..."

but

The USA edition on its corresponding page (also page 1050) says in part: "... Bibliography ... modern technology and innovation have vastly improved the capabilities for disseminating and retrieving information on a global scale. Specifically, there is the introduction of arXiv.org, an online repository where physicists ... can publish preprints (or 'e-prints') of their work before (or even instead of!) submitting it to journals. ...as a consequence the pace of research activity has accelerated to an unprecedented (or, as some might consider, an alarming) degree. ...". However, **the USA edition omits the laudatory reference to Paul Ginsparg that is found in the UK edition**.

For another example:

The USA edition adds some additional references, including (at page 1077): "... **Pitkanen, M.** (1994). p-Adic description of Higgs mechanism I: p-Adic square root and p-adic light cone. [hep-th/9410058] ...".

Note that Matti Pitkanen was in 1994 allowed to post papers on the e-print archives now known as arXiv (obviously including the paper referenced immediately above), but since that time **Matti Pitkanen has been blacklisted by arXiv** and is now barred from posting his work there. His web page account of being blacklisted is at http://www.physics.helsinki.fi/~matpitka/blacklist.html .

It seems to me that it is likely that the omission of praise of arXiv's Paul Ginsparg and the inclusion of a reference to the work of now-blacklisted physicist Matti Pitkanen are deliberate editorial decisions.

Also, since the same phrase "... physicists ... can publish preprints (or 'e-prints') of their work before (or even instead of!) submitting it to journals. ..." appears in both editions, it seems to me that Roger Penrose favors the option of posting on arXiv without the delay (and sometimes page-charge expense) of journal publication with its refereeing system.

I wonder what events between UK publication on July 29, 2004 and USA publication on February 22, 2005 might have influenced Roger Penrose to make the above-described changes in the USA edition ?

There are two possibly relevant events in that time frame of which I am aware:

- The appearance around November 2004 of the ArchiveFreedom web site, which web site documents some cases of arXiv blacklisting etc;
- According to a CERN web page at http://documents.cern.ch/EDS/current/access/action.php?doctypes=NCP "... CERN's Scientific Information Policy Board decided, at its meeting on the 8th October 2004, to close the EXT-series. ...". Note that the CERN EXT-series had been used as a public repository for their work by some people (including me) who had been blacklisted by arXiv .

Tony Smith's Home Page