Scaling of Volumes used in CERN CDS preprint EXT-2003-087 Calculations:

Calculations of force strengths and particle masses in my CERN CDS preprint EXT-2003-087 use volumes of geometric objects whose scale is fixed at unit radius. A detailed justification for this is found in "Scaling of Wyler's Expression for alpha", by Robert Gilmore, Phys. Rev. Lett. 28 (1972) 462-464:

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Scaling of Wyler's Expression for α

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We present the reason for setting $r=1$ in Robertson's recent discussion of Wyler's expression for the fine-structure constant. Only for the choice $r=1$ are all elements in the bounded domain D^* also coset representatives and therefore group operations.

Wyler's[1,2] expression for the fine-structure constant has been discussed in a recent Letter by Robertson.[3] Several objections to this expression were raised, the principal one being based on a scaling argument. We present the mathematical reason for fixing the scale at $r=1$.

To place the scaling argument in perspective, we review the relationship between some simple groups and their associated symmetric spaces.[4]

In the defining 3×3 matrix representation, the Lie algebra L[SO(3)] for the special orthogonal group SO(3) consists of real antisymmetric 3×3 matrices. These matrices can be written as a direct sum of matrices:

$$\left[\begin{array}{cc|c} 0 & a & b \\ -a & 0 & c \\ \hline -b & -c & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & a & 0 \\ -a & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \oplus \left[\begin{array}{cc|c} 0 & 0 & b \\ 0 & 0 & c \\ \hline -b & -c & 0 \end{array}\right], \qquad (1)$$

the first matrix being an element of the Lie subalgebra $\mathfrak{k} = L[SO(2)]$, and the second an element of its orthogonal complement[4]

$$\mathfrak{p} = L[SO(3)] \underset{(\mathrm{mod})}{-} L[SO(2)].$$

The subspace $\mathfrak{k}$ of real antisymmetric 2×2 matrices maps onto the Lie subgroup SO(2) under the exponential mapping. The orthogonal complementary subspace $\mathfrak{p}$, which is not closed under commutation, maps under the exponential mapping onto the space of coset representatives:

$$L[SO(3)] - L[SO(2)] \xrightarrow{\exp} SO(3)/SO(2), \qquad (2)$$

where

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix} \frac{\sin(b^2+c^2)^{1/2}}{(b^2+c^2)^{1/2}}, \qquad (4a)$$

$$z = \cos(b^2+c^2)^{1/2} = \left[I_1 - (x, y)\begin{pmatrix} x \\ y \end{pmatrix}\right]^{1/2}. \qquad (4b)$$

There is a one-to-one correspondence between the coset representatives in the space SO(3)/SO(2) and the points on the surface of the sphere

$$z^2 + [x^2 + y^2] = 1 \qquad (5)$$

which follows directly from Eqs. (4). Of course, it is true that there is also a one-to-one correspondence between the space SO(3)/SO(2) and the surface

$$x^2 + y^2 + z^2 = a^2, \quad a \neq 1.$$

But we have shown that Eq. (5) arises naturally. The "scaling parameter" a is fixed uniquely at 1 by the condition that the matrices on the right of Eq. (3) are elements of the orthogonal group SO(3), for which the sum of the squares of the elements in any row or column is 1.

The coset representatives of SO(3)/SO(2) are those 3×3 matrices in the group SO(3) whose structure is given explicitly on the right-hand side of Eq. (3). Since these matrices can be uniquely constructed from the upper right-hand 2×1 submatrix col(x, y), these submatrices will also be called coset representatives.

The noncompact coset space $SO(n, 2)/SO(n)SO(2)$ can be treated in an entirely analogous manner. For the sake of simplicity, we first treat the coset space SO(2, 1)/SO(2). In a calculation entirely analogous to Eqs. (1)–(4), the mapping

$$L[SO(2,1)] - L[SO(2)] \xrightarrow{\exp} SO(2,1)/SO(2)$$

ping onto the space of coset representatives:

$$L[SO(3)] - L[SO(2)] \xrightarrow{\exp} SO(3)/SO(2). \qquad (2)$$

This mapping is given by

$$\left[\begin{array}{cc|c} 0 & 0 & b \\ 0 & 0 & c \\ \hline -b & -c & 0 \end{array}\right] \xrightarrow{\exp} \left[\begin{array}{c|c} \left[I_2 - \begin{pmatrix} x \\ y \end{pmatrix}(x, y)\right]^{1/2} & \begin{matrix} x \\ y \end{matrix} \\ \hline -x \quad -y & z \end{array}\right], \qquad (3)$$

tirely analogous to Eqs. (1)–(4), the mapping

$$L[SO(2,1)] - L[SO(2)] \xrightarrow{\exp} SO(2,1)/SO(2)$$

is given by

$$\left[\begin{array}{cc|c} 0 & 0 & b \\ 0 & 0 & c \\ \hline b & c & 0 \end{array}\right] \xrightarrow{\exp} \left[\begin{array}{c|c} \left[I + \begin{pmatrix} x \\ y \end{pmatrix}(x, y)\right]^{1/2} & \begin{matrix} x \\ y \end{matrix} \\ \hline x \quad y & z \end{array}\right], \qquad (6)$$

where

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix} \frac{\sinh(b^2+c^2)^{1/2}}{(b^2+c^2)^{1/2}},$$
$$z = \cosh(b^2+c^2)^{1/2} = \left[I_1 + (x, y)\begin{pmatrix} x \\ y \end{pmatrix}\right]^{1/2}. \qquad (7)$$

The coset representatives are in one-to-one correspondence with the unbounded set of points (x, y) (the x-y plane of Fig. 1). The coset representatives are also in one-to-one correspondence with the points on the upper sheet ($z \geq 1$) of the two-sheeted hyperboloid

$$z^2 - [x^2 + y^2] = 1. \qquad (8)$$

Equation (8), like Eq. (5), comes directly from the metric-preserving property defining the group SO(2, 1). Under the action of the group SO(2, 1), the transformation properties of the coset representatives (x, y) are not at all simple. Coset representatives with simple, analytic,[5-9] fractional linear transformation properties are related to the representatives (x, y) by a simple projective transformation. The projective transformation

$$\left[\begin{array}{c|c} & x \\ & y \\ \hline x \quad y & z \end{array}\right] \to \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix} z^{-1} \qquad (9)$$

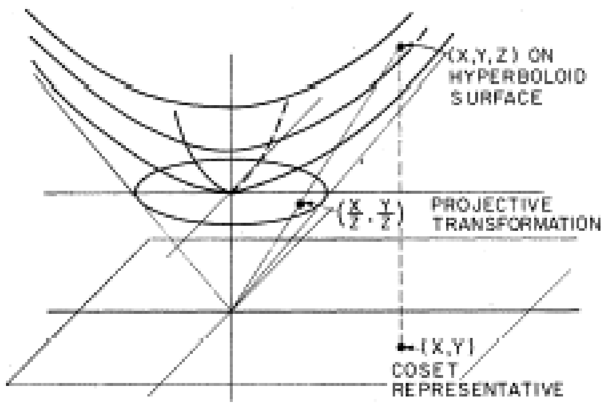

FIG. 1. Representatives of the cosets in the space SO(2, 1)/SO(2) can be chosen as the points (x,y), the points (x,y,z) on the upper sheet of the two-sheeted hyperboloid, or the projectively related points $(x,y)/z$ in the interior of the unit circle. The hyperboloid effects a transformation between the unbounded and bounded domains.

maps the coset representatives into the interior of a bounded domain D^1 determined by

$$(x/z)^2 + (y/z)^2 < 1. \qquad (10)$$

The relations between these various domains is shown in Fig. 1.

The domain D^n is obtained from the unbounded domain of coset representatives

$$\exp\left[\begin{array}{c|c} & A \\ \hline +A^\dagger & \end{array}\right] = \left[\begin{array}{c|c} [I_n + XX^\dagger]^{1/2} & X \\ \hline +X^\dagger & [I_2 + X^\dagger X]^{1/2} \end{array}\right] \in \frac{SO(n,2)}{SO(n)SO(2)}, \qquad (11)$$

$$X = A\frac{\sinh(A^\dagger A)^{1/2}}{(A^\dagger A)^{1/2}} \quad \text{(real } n\times 2 \text{ matrix)}, \qquad (12)$$

by performing a projective[5-9] transformation

$$X \to Z = X[I_2 + X^\dagger X]^{-1/2}. \qquad (13)$$

For this reason, the scaling parameter[3] r is fixed at 1. Only for the choice $r = 1$ are all elements in the bounded domain D^n also coset representatives and therefore group operations.

Once again, the coset representatives of $SO(n, 2)/SO(n)SO(2)$ are those real $(n+2)\times(n+2)$ matrices in the group $SO(n, 2)$ whose structure is indicated explicitly on the right-hand side of Eq. (11). Since trices Z are also called coset representatives.

We remark that the factor 4π which appears in Wyler's expression is the invariant volume of the coset space SO(3)/SO(2) (see below). The group SO(3) acts in the real Euclidean three-dimensional part of physical space-time. Robertson correctly leaves this factor unscaled. Thus, the scaling argument has been carried out inconsistently: Either *all* coset volumes appearing in the calculation should be scaled [$4\pi \to 4\pi a^2$ as well as Eqs. (9′)–(11′) of Ref. 3, $r \neq a$], or all should remain unscaled. In fact, no volumes are scaled.

A second objection[3] is that $V(D^n)$ and $V(Q^n)$ are

... in the group SO(n, 2) whose structure is indicated explicitly on the right-hand side of Eq. (11). Since these matrices can be uniquely reconstructed from the upper right-hand $n\times 2$ submatrix X, these submatrices X are also called coset representatives. The $(n+2)\times(n+2)$ matrices can also be reconstructed from the $n\times 2$ matrices Z using $X = Z[I_2 - Z^\dagger Z]^{-1/2}$. For this reason, the $n\times 2$ ma-

... main unscaled. In fact, no volumes are scaled.

A second objection[3] is that $V(D^n)$ and $V(Q^n)$ are not invariant volumes of the unbounded coset domains SO(n, 2)/SO(n)SO(2), which diverge, but rather the Euclidean volumes of the projectively related bounded domains, which are finite. Using the finite Euclidean volumes rather than the infinite invariant volumes, in analogy with Bethe's

dropping of an infinite correction term to obtain a finite result, may be another instance in which the replacement of a divergent value by a finite value can lead to a well-defined and significant result. The occurrence of the Euclidean volumes $V(Q^n)$ and $V(D^n)$ should be considered a strong point of Wyler's result, rather than an objectionable feature. These volumes arise naturally as the normalizing coefficients in the Poisson and Bergman kernels, which are reproducing functions and are defined in a nonlinear way.[7] The Poisson kernel is the image of a space-time scalar Green's function, when both arguments of the kernel are on the boundary Q^n of D^n ($n = 4, 5$). These propagators are for an $m=0$ particle ($n=4$), and an $m\neq 0$ particle ($n=5$). The usual[10] space-time scalar Green's function ($m=0$) is defined, for the sake of subsequent simplicity, with a factor 4π multiplying the δ-function singularity, while the $m\neq 0$ propagator is defined without[11] this additional factor. It is this factor of 4π which enters Wyler's expression[1-3] for the fine-structure constant (see above). Unfortunately, Wyler does not clearly specify how the Bergman density $V^{-1}(D^5)$ enters the calculation. This is, in fact, still an unanswered[12] question. There is as yet no rigorous mathematical basis for the appearance of the factor $[V(D^5)]^{1/4}$. At best, only a heuristic argument can be advanced.

Wyler's work has pointed out that it is possible to map an unbounded physical domain—the interior of the forward light cone—onto the interior of a bounded domain on which there also exists a complex structure. This mapping should prove of immense calculational value in the future. This transformation from unbounded to bounded complex domains is mathematically rigorous, and is valid whether or not the fine-structure constant can in fact be calculated following Wyler's heuristic arguments.

Although serious objections can be raised about Wyler's calculation of the fine-structure constant, the scaling argument[3] is not one of them.

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[1] A. Wyler, C. R. Acad. Sci. A269, 743 (1969).
[2] A. Wyler, C. R. Acad. Sci. A272, 186 (1971).
[3] B. Robertson, Phys. Rev. Lett. 27, 1545 (1971).
[4] S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic, New York, 1962).
[5] E. Cartan, Abh. Math. Sem. Univ. Hamburg 11, 116 (1935); see especially Sect. 40.41.
[6] C. L. Siegel, "Analytic Functions of Several Complex Variables," lecture notes, 1948–1949, Institute for Advanced Study, Princeton, N. J.; see especially pp. 152–161, 166–174.
[7] L. K. Hua, *Harmonic Analysis and Functions of Several Complex Variables in the Classical Domains*, Translations of Mathematical Monographs (American Mathematical Society, Providence, R. I., 1963), Vol. 6. See expecially Eqs. (4.3.1), (4.3.6), (4.3.10), (4.4.1)–(4.4.3), pp. 83–86.
[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, New York, 1969), Vol. II, p. 161.
[9] A. Wyler, Arch. Ration. Mech. Anal. 31, 35 (1968), Eqs. (7.8), (7.10).
[10] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962). See especially Eq. (6.55), p. 183.
[11] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1956), Appendix A1, pp. 419–424.
[12] Private communication from I. M. Singer, 1971.

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