288 Sparks of Genesis
and Boundary-Bulk Physics

According to Bernard Pick's 1913 work The Cabala on a sacred-texts.com web page:

"... the Zohar is ... [a] production ... of ... thirteenth century ... Spain ... by Moses de Leon (1250-1305) ...".

According to web pages by Rabbi Moshe Miller about Arizal on the www.safed-kabbalah.com web site:

"... Rabbi Yitzchak Luria (the Arizal) ... 1534-1572 c.e. ... set out to explain ... the kabbalistic literature ... particularly Zohar ...
There are five areas of focus in the Arizal's teachings ... :

- the concept of **tzimtzum (G-d's self-contraction**, so to speak) through its various stages ...
- **Prior to creation, there was only G-d and His infinite revelation of Himself, the Or Ein Sof**, filling all existence ...[corresponding to

the letter א Aleph (first of the Hebrew alphabet) placed before the First Verse of the First Book Genesis of the Torah
and
the dimensionless Empty Set ...]

the tzimtzum ... established a radical distinction between Creator and created (from the viewpoint of the created, although not from the viewpoint of the Creator ...), ... **so that creation comes about by way of a "quantum leap"** ...
[corresponding to

the first word of the First Verse בְּרֵאשִׁית that begins with Genesis

letter number 1 בְּ Bet and ends with Genesis letter number 6 ת Tav

and

בְּ the 0-dimensional Natural Numbers created from the Empty Set by the Peano unitizer operation 0 -> {0} = 1 as described by David Finkelstein in his book "Quantum Relativity" (Sprimger 1996)

רֵ the 1-dimensional Real Numbers created from the Natural Numbers by ratios and completion

א the 2-dimensional Algebraically Complete Complex Numbers created from the Real Numbers by Cayley-Dickson Doubling

שִׁ the 4-dimensional Associative NonCommutative Quaternions created from the Complex Numbers by Cayley-Dickson Doubling

י the 8-dimensional Alternative NonAssociative Octonions created from the Quaternions by Cayley-Dickson Doubling

ת the 16-dimensional Sedenions created from the Quaternions by Cayley-Dickson Doubling

- the process of **shevirat hakeilim (the shattering of the vessels in the world of Tohu)** ...
- The first "world" (plane of existence) that came into being after the tzimtzum is called Adam Kadmon. ... the light in Adam Kadmon ... manifested as ... sefirot [that] compose the world of Tohu (chaos or disorder) ... Due to the intensity and exclusivity of the lights ... the vessels of the lower sefirot of Tohu shattered ...[corresponding to the First Verse

א. בְּרֵאשִׁית, בָּרָא אֱלֹהִים, אֵת הַשָּׁמַיִם, וְאֵת הָאָרֶץ

1 In the beginning God created the heaven and the earth.

that ends with Genesis letter number 28 ץ Zadi-final

and

the shattering of Division Algebra structure at the formation of the **Sedenions, which have non-trivial Zero Divisors** is described by Guillermo Moreno in arXiv math/0512517 as "... **For ...[Sedenions]... the set of zero divisors ... of fixed norm** can be identified with V7,2 the real Stiefel Manifold of two frames in R7 and the singular set of (x,y) with xy = 0 and ||x|| = ||y|| = 1 **is homeomorphic to G2** the exceptional simple Lie group of rank 2. ...". If the x and y coordinates are allowed to expand/contract from unit norm, then the set of Zero Divisors can be seen as having **two more dimensions, equivalent to two scalar dimensions**, so that **the Full Zero Divisors of Sedenions have 16 dimensions equivalent to R + G2 + R .** (see also the descriptions by Robert P. C. de Marrais in his papers including arXiv 0804.3416)

Consider the original expansion of our universe as emerging from an initial origin point in a 4-dimensional D4+ HyperDiamond lattice Spacetime.
Consider that the duration of the initial Zizzi Inflation Era is marked (non-linearly) by 22 shells,one for each of the 22 letters of the Hebrew alphabet, each shell containing the number of vertices in the corresponding D4+ lattice shells.
Then, the first letter corresponds to the first shell with 8 vertices.
The shell corresponding to the 21st letter has 256 vertices, the maximum number of elements in the Cl(8) Clifford algebra and in the Odu of IFA.
The shell corresponding to the 22nd (last) letter of the Hebrew alphabet should correspond to the shattering of the vessels, and is the first shell to exceed 256 vertices. It has 288 vertices. These are referred to by the Arizal as **the 288 nitzotzin ("sparks") - the initial number of fragments from the vessels that broke**. The ... process is alluded to in .... [Genesis 1 : 2 from www.mechon-mamre.org web site

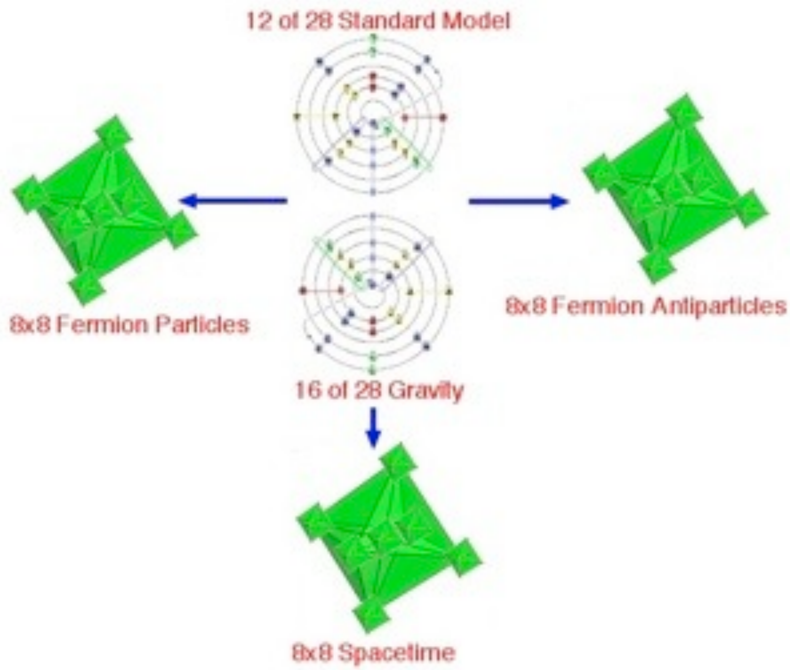
ב וְהָאָרֶץ, הָיְתָה תֹהוּ וָבֹהוּ, וְחֹשֶׁךְ, עַל-פְּנֵי תְהוֹם; וְרוּחַ אֱלֹהִים, מְרַחֶפֶת עַל-פְּנֵי הַמָּיִם.

2 Now the earth was unformed and void, and darkness was upon the face of the deep; and the spirit of God hovered over the face of the waters.

with the central three letters (Genesis numbers 68, 69, and **70**) of the word "hovered" indicated by a red box. Note that **70** is the dimensionality of the central part of the graded structure of Cl(8)

$$16x16 = 256 = \mathbf{1} + 8 + \mathbf{28} + 56 + \mathbf{70} + 56 + \mathbf{28} + 8 + \mathbf{1}$$

whose 256 elements were the largest number of shell vertices that were coherently together prior to the shattering, and note that Cl(8) contains the E8 Lie algebra



and that 248-real-dim E8 = 120-real-dim D8 + 128-real-dim D8 half-spinor
and that

120-dim D8 =	**128 half-spinor of D8 =**
= Spacetime 64	**= Particle 64**
+ Gravity 28	**+ AntiParticle 64**
+ Standard Model 28	

so that E8 whose graded structure is

$$8 + 28 + 56 + 64 + 56 + 28 + 8$$

describes a realistic Physics Lagrangian:

$$\textbf{Lagrangian:} \quad \int_{\text{KKspacetime}} \text{gauge term} + \text{fermion term}$$

but does not by itself describe Quantum Interactions.]...

The Arizal explains that the word ... hovered (merachefet) ... is actually a compound of two words: met and rapach - signifying that 288 (the numerical value of rapach) fragments had died (met) ...[in]... the shattering of the vessels of Tohu into 288 initial sparks ... The shattering of the sefirot of Tohu ... serves a very specific and important purpose, which is to bring about a state of separation or partition of the light into distinct qualities and attributes, and thereby introduce diversity and multiplicity into creation ...".

- the **Tikkun (rectification)** of that shevira through birur hanitzotzot **(elevating the sparks)** ...
  [The]... process of extracting the sparks is called birur, which is part of a larger cosmic plan called **Tikkun - rectification or restoration of the broken vessels** ... When the sparks ... are rebuilt into the vessels of Tikkun ... the repaired vessels will be able to contain the light ...

- the concept of **partzufim ... compound structures ... in arrays that interact with each other**

... the partzufim are compound structures of the sefirot ... In the universe of partzufim, it may be said that the chief dynamic of creation is not evolution (hishtalshelut), but rather interaction (hitlabshut). ...[Therefore,
the Partzufim is the not the E8 Lie Algebra
but must be an algebra of Quantum Interactions among the E8 Lagrangian structures.
E8 can be contracted to D8 = Spin(16) and then further contracted to A7 = SU(8) leading to a 248-dim H248 based on nilpotent Heisenberg algebra structure that acts naturally as an Algebra of Quantum Interactions among the E8 Lagrangian structures:

$$H248 = SU(8) + h\_92$$

h_92 is a 185-dimensional Heisenberg Lie algebra
for 92 sets of creation-annihilation operators:

64 Fermion Particle Creators + 64 Fermion AntiParticle Creators

28 Gravity Boson Creators + 28 Standard Model Boson Creators

plus

1 central Heisenberg Algebra element

The 63 generators of SU(8) plus the 1 central Heisenberg Algebra element represent the 8x8 Position/Momentum combinations of 8-dim Spacetime.

H248 lives in the Quantum Bulk has graded structure

28 + 64 + (1+63) + 64 + 28

and
when combined with the Lagrangian E8 with graded structure

8 + 28 + 56 + 64 + 56 + 28 + 8

gives

a Quantum Theory that is equivalent to Path Integral Quantization of the Lagrangian Theory.

Together, **E8 and H248 form a 496-dimensional Partzufim**.

Note that **0, 1, 6, and 28 are Perfect Numbers**, and that the only others below 33,550,336 (related to the Mersenne Prime 8,191 = 2^13 -1) are

the Perfect Numbers
496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248
(related to the Mersenne Prime 31 = 2^5 -1)
and
8,128 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1,016 + 2,032 + 4,064
(related to the Mersenne Prime 127 = 2^7 -1).

Thus,
the E8 + H248 Partzufim
at the very Beginning of the Inflationary Expansion of Our Universe
(a non-unitary process due to the non-unitarity of nonassociative Octonions)
corresponds to the Perfect Number 496,

which is consistent with Genesis letter number 496 being the last שְׁ Shin in

Chapter 1 Verse 11

ח וַיִּקְרָא אֱלֹהִים לָרָקִיעַ, שָׁמָיִם; וַיְהִי-עֶרֶב וַיְהִי-בֹקֶר, יוֹם שֵׁנִי. {פ}	**8** And God called the firmament Heaven. And there was evening and there was morning, a second day. **{P}**
ט וַיֹּאמֶר אֱלֹהִים, יִקָּווּ הַמַּיִם מִתַּחַת הַשָּׁמַיִם אֶל-מָקוֹם אֶחָד, וְתֵרָאֶה, הַיַּבָּשָׁה; וַיְהִי-כֵן.	**9** And God said: 'Let the waters under the heaven be gathered together unto one place, and let the dry land appear.' And it was so.
י וַיִּקְרָא אֱלֹהִים לַיַּבָּשָׁה אֶרֶץ, וּלְמִקְוֵה הַמַּיִם קָרָא יַמִּים; וַיַּרְא אֱלֹהִים, כִּי-טוֹב.	**10** And God called the dry land Earth, and the gathering together of the waters called He Seas; and God saw that it was good.
יא וַיֹּאמֶר אֱלֹהִים, תַּדְשֵׁא הָאָרֶץ דֶּשֶׁא עֵשֶׂב מַזְרִיעַ זֶרַע, עֵץ פְּרִי עֹשֶׂה פְּרִי לְמִינוֹ, אֲשֶׁר זַרְעוֹ-בוֹ עַל-הָאָרֶץ; וַיְהִי-כֵן.	**11** And God said: 'Let the earth put forth grass, herb yielding seed, and fruit-tree bearing fruit after its kind, wherein is the seed thereof, upon the earth.' And it was so.

Shin שֵׁ looks like a Tree of Life that Grows, Multiplies by its Seeds, and Evolves. Compare page 135 of Jung's Red Book in Cap. xix "The Gift of Magic"

where Jung says (English translation footnote 248): "Completed on 25 November 1922. The fire comes out of Muspilli and grasps the tree of life. A cycle is completed, but it is the cycle within the world egg. A strange God, the unnameable God of the solitary, is incubating it. New creatures form from the smoke and ashes."

Since **Perfect Number 496 corresponds to the Beginning of Inflation**, it seems that **Perfect Number 8,128 should correspond to the End of Inflation**, and therefore indicate the duration of Inflation, the fundamental unit of mass, and the number of particles created during Inflation.

Genesis letter number 8,128 is the fourth ע Ayin in Chapter 7 Verse 4, that is, the word 40 of 40 days in the statement

"... I will cause it to rain upon the earth 40 days ... "
which rain (see verse 23): "... blotted out every living substance ... from the earth; and Noah only was left, and they that were with him in the ark ...".

The 2 branches of ע Ayin look like a fork in the Road of History leading to two alternative futures:

the Death Future of most of Life on Earth and **the Life Future of the Ark beings**.

Note that 8,128 = 64 x 127 = 64 x (128 - 1).

As Paola Zizzi said in gr-qc/0007006: "... during inflation, the universe can be described as a superposed state of quantum ... [qubits]. The self-reduction of the superposed quantum state ... reached at the end of inflation ... corresponds to a superposed state of ... [$10^{19} = 2^{64}$ qubits] ... ...[at]... the decoherence time ... [$T_{decoh} = \sqrt{10^{19}}\ T_{planck} = 10^{-34}$ sec]. ...",
so **2^{64} tells us the duration of Inflation**.
2^{64} qubits corresponds to the Clifford algebra Cl(64) = Cl(8x8).
By the periodicity-8 theorem of real Clifford algebras that
Cl(K8) = Cl(8) x ... tensor product K times ... x Cl(8),
we have: Cl(64) = Cl(8x8) =
= Cl(8) x Cl(8) x Cl(8) x Cl(8) x Cl(8) x Cl(8) x Cl(8) x Cl(8)

Therefore, Cl(64) is the first (lowest dimension) Clifford algebra at which we can reflexively identify each component Cl(8) with a vector in the Cl(8) vector space. This reflexive identification/reduction causes decoherence.

It is the reason that our universe decoheres at N = 2^64 = 10^19
which Decoherent Collapse into the Many Worlds
of the Many-Worlds Quantum Theory

led to our World being only one of the Many.

At the time Tdecoh = 10^(-34 sec) at the End of Inflation, the number of qubits is Ndecoh = 10^19 = 2^64 .

Each qubit at the end of inflation corresponds to a Planck Mass Black Hole, which undergoes decoherence and, in a process corresponding to Reheating in the Standard Inflationary Model, each qubit transforms into 2^64 = 10^19 elementary first-generation fermion particle-antiparticle pairs.

The resulting 2^64 x 2^64 = 2^128 = 10^19 x 10^19 = 10^38 fermion pairs populating the Universe Immediately After Inflation constitutes a Zizzi Quantum Register of order n_reh = 10^38 = 2^128.

Since, as Paola Zizzi says in gr-qc/0007006, (with some editing by me denoted by [ ]): "... the quantum register grows with time. ... At time Tn = (n+1) Tplanck the quantum gravity register will consist of (n+1)^2 qubits. [Let N = (n+1)^2] ...", we have the number of qubits at Reheating:

Nreh = (n_reh)^2 = (2^128)^2 = 2^256 = 10^77

Since each qubit at Reheating should correspond, not to Planck Mass Black Holes, but to fermion particle-antiparticle pairs that average about 0.66 GeV, we have the result that the number of particles in our Universe at Reheating is about 10^77 nucleons, so **(2^128)^2 tells us the number particles in our universe**.

After Reheating, our Universe enters the Radiation-Dominated Era, and, since there is no continuous creation, particle production stops, so the 10^77 nucleon Baryonic Mass of our Universe has been mostly constant since Reheating, and will continue to be mostly constant until Proton Decay.

The present scale of our Universe is about R(tnow) = 10^28 cm, so that its volume is now about 10^84 cm^3, and its baryon density is now about 10^77 protons / 10^84 cm^3 = 10^(-7) protons/cm^3 = 10^(-7-19-5) gm / cm^3 = 10^(-31) gm / cm^3 = roughly the baryonic mass density of our Universe.

Since the critical density of our Universe is about 10^(-29) gm / cm^3, it is likely that the excess of the critical mass of our Universe over its baryonic mass is due to a cosmological constant as described by Conformal Gravity in the E8 Physics model which gives a ratio of
Dark Energy : Dark Matter : Ordinary Matter of 0.753 : 0.202 : 0.045 .]...

the nature of the soul, the purpose of its descent into this world, and its relationship with the higher realms and ultimately with G-d. ...

the soul is both part of the Creator and at the same time it is created -
its luminous essence is "a tiny spark of G-dliness,"
and the sheath in which it is clothed is a created being,
albeit a spiritual being and not physical.
As the soul emanates from the Ein Sof - the Infinite One -
eventually to be clothed in the physical body ...
Thus man is a microcosm of creation and his actions have cosmic significance ...
He is able to affect the balance of the universe,
both spiritual and physical,
by his kavanot (mystical intentions) and yichudim (unifications of the sefirot). ...".

Since Ndecoh = 2^64 = 10^19 qubits is just an order of magnitude larger
than the number of tubulins Ntub = 10^18 of the human brain,
and
Conscious Thought is due to superposition states of those 10^18 tubulins,
and
since a brain with Ndecoh = 10^19 tubulins would undergo self-decoherence
and would therefore not be able to maintain the superposition necessary for thought,
it seems that the human brain is about as big as an individual brain can be.
The Zizzi Self-Decoherence can be compared to GRW decoherence.
Thus

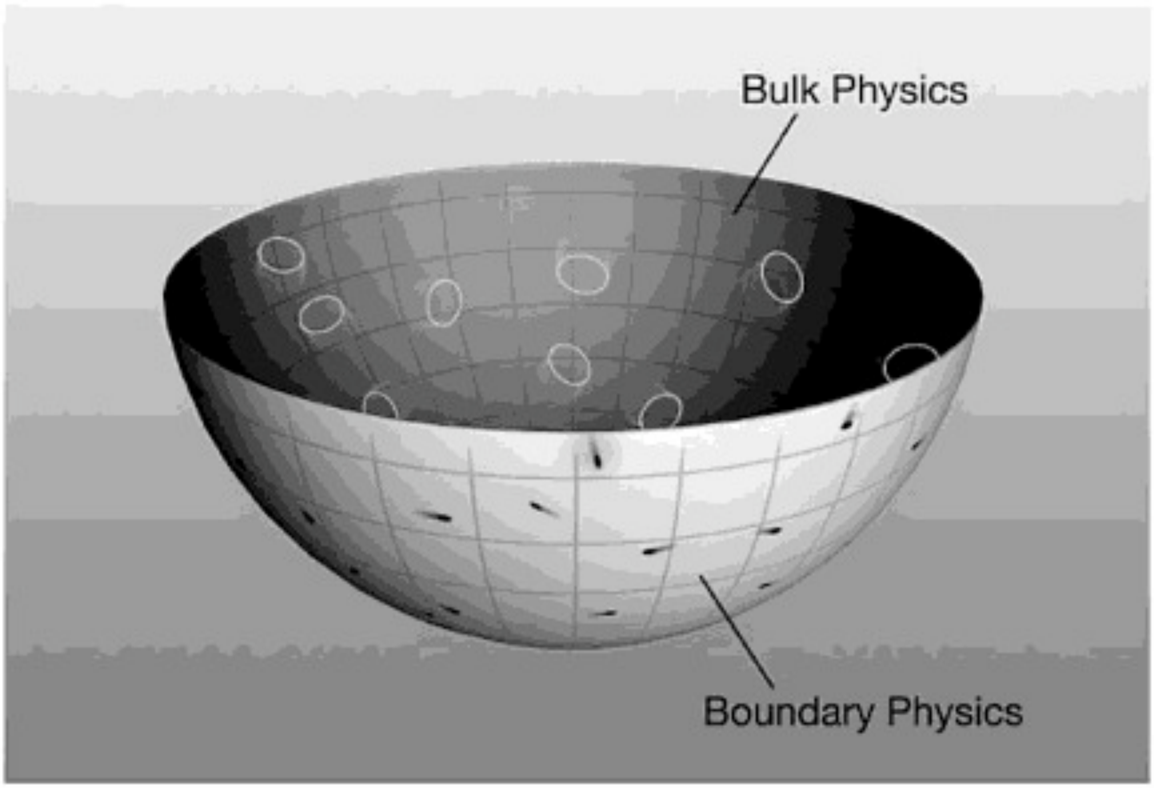
the Mind of Man seems to be an image of the Mind of Our Universe.

Boundary-Bulk Physics Table of Contents:

Physics: Real Boundary - Quantum Bulk

Frank Dodd (Tony) Smith, Jr. - 2011

Brian Greene in his book "The Hidden Reality" (Knopf 2011) describes
"... an explicit link between physics taking place in a region ...



... and physics taking place on that region's boundary ... an explicit realization of holography. That's the basic idea. ...".
Brian Greene chooses to identify the Boundary-bulk physics as:

Boundary - Quantum Field Theory - "... four-dimensional supersymmetric conformally invariant quantum field theory ...".
Bulk - Superstring Theory - "... ten-dimensional string theory on AdS5xS5 ..."

On the other hand, I do not think that conventional supersymmetry is realistic, so I reject Brian Greene's choices for Bulk - Boundary physics and propose:

Real Boundary - their Shilov Boundaries - E8 Real Lagrangian Structure

Quantum Bulk - Bounded Complex Domains - H248 Quantum Structure

Real Boundary E8 Lagrangian

Lagrangian: $\int_{\text{KKspacetime}}$ gauge term + fermion term

8 Fundamental Fermion Particles (generations 2 and 3 emerge at low energies) are represented by a copy of the 8-real-dimensional Shilov Boundary RP1 x S7 of the 8-complex-dimensional Complex Bounded Domain that corresponds to the 16-real-dimensional Symmetric Space Spin(10) / Spin(8)xU(1) that by Triality represents 8s and 8c half-spinors as well as 8v of 8-dim Spacetime. Each of 8p Particles has 8cp components for a total of 8px8cp = 64.

8 Fundamental Fermion AntiParticles (generations 2 and 3 emerge at low energies) are represented by a copy of the 8-real-dimensional Shilov Boundary RP1 x S7 of the 8-complex-dimensional Complex Bounded Domain that corresponds to the 16-real-dimensional Symmetric Space Spin(10) / Spin(8)xU(1) that by Triality represents 8c and 8s half-spinors as well as 8v of 8-dim Spacetime. Each of 8p AntiParticles has 8cp components for a total of 8px8cp = 64.

8-dimensional Spacetime (reduces to M4 x CP2 Kaluza Klein at low energies) is a copy of the 8-real-dimensional Shilov Boundary RP1 x S7 of the 8-complex-dimensional Complex Bounded Domain that corresponds to the 16-real-dimensional Symmetric Space Spin(10) / Spin(8)xU(1). There are 8px8m = 64 combinations of 8pPosition/8mMomentum.

Conformal MacDowell-Mansouri Gravity Gauge Term uses 16 of the 28 generators of a copy of Spin(8).
Standard Model Term gets SU(3) from 16 of the 28 generators of a copy of Spin(8) and SU(2)xU(1) from the local symmetry of the CP2 of Kaluza-Klein.

These Real Components of the Lagrangian combine to form a copy of
248-real-dim E8 = 120-real-dim D8 + 128-real-dim D8 half-spinor:

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 120-dim D8 =
= Spacetime 64
+ Gravity 28
+ Standard Model 28
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128 half-spinor of D8 =
 = Particle 64
 + AntiParticle 64
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Quantum Bulk H248 Structure

The Complex Quantum Bulk has twice the real dimensionality
of its Real Lagrangian Boundary, so look at a second copy of E8.
E8 can be contracted to D8 = Spin(16) and then further contracted to A7 = SU(8)
leading to the structure

H248 = SU(8) + h_92

h_92 is a 185-dimensional Heisenberg Lie algebra
for 92 sets of creation-annihilation operators:

64 Fermion Particle Creators + 64 Fermion AntiParticle Creators

28 Gravity Boson Creators + 28 Standard Model Boson Creators

plus

1 central Heisenberg Algebra element

The 63 generators of SU(8) plus the 1 central Heisenberg Algebra element
represent the 8px8m Position/Momentum combinations of 8-dim Spacetime.

H248 lives in the Quantum Bulk has graded structure

28 + 64 + (1+63) + 64 + 28

and
when combined with the Lagrangian E8 with graded structure

8 + 28 + 56 + 64 + 56 + 28 + 8

gives

a Quantum Theory that is equivalent to Path Integral Quantization of the Lagrangian Theory.

References for Boundary-Bulk Constructions:

Structure of H248 = SU(8) x h_92:
Rutwig Campoamor-Stursberg in Acta Physica Polonica B 41 (2010) 53-77 said
"... We have classified all contractions of complex simple exceptional Lie algebras onto semidirect products of semisimple and Heisenberg algebras.
An analogous procedure holds for the real forms of the exceptional algebras ...
Semidirect products s + h_N of semisimple and Heisenberg Lie algebras for N independent sets of boson creation and annihilation operators ... constitute ...[a]... tool to combine inner and outer symmetries of physical systems. ...
Contractions of E8 ... E8 contains D8 contains A7 ... N = 92
... This reduction gives rise to the contraction
...[248-dim E8 -> A7 + h_92 = 63-dim SU(8) + 185-dim h_92 = H248]...".

Fermion as well as boson creation-annihiliation operators:
Pierre Ramond in hep-th/0112261 said:
"... the coset F4 / SO(9) ... is the sixteen-dimensional Cayley projective plane ...
[represented by]... the SO(9) spinor operators [which] satisfy Bose-like commutation relations ... Curiously,
if ...[the scalar and spinor 16 of F4 are both]... anticommuting,
the F4 algebra is still satisfied ...".
The same reasoning applies to E8 and the 64 + 64 = 128-dim half-spinor of D8.

Physical interpretation of SU(8):
Real Boundary Lagrangian E8 has graded structure

8s + 28 + 56 + 64 + 56 + 28 + 8c

in which 8s and 8c are half-spinor representations of D4 = Spin(8).
Cl(8) Clifford Algebra has graded structure

1 + 8v + 28 + 56 + (70 = 8s + 6 + 48 + 8c) + 56 + 28 + 8v* + 1

in which 8v is the D4 = Spin(8) vector position representation of 8-dim spacetime and 8v* is the is the dual momentum representation of 8-dim spacetime.
By D4 = Spin(8) Triality and Position-Momentum Duality the four entities 8s , 8c , 8v , 8v* are all isomorphic and effectively interchangeable, so that E8 effectively lives inside Cl(8) and
the 8v and 8v* of Cl(8) can represent E8 8-dim position and 8-dim momentum.
The exterior algebra underlying the Cl(8) Clifford Algebra has graded structure

1 + 8v + 28 + 56 + 70 + 56 + 28 + 8v* + 1

so that 64-dim U(8) = SU(8) x U(1) represents the tensor product 8v x 8v*
and 63-dim SU(8) represents 8-dim spacetime position and momentum combinations that have unit determinant.

Position-Momentum Duality:
Dennis W. Marks in his paper A Binary Index Notation for Clifford Algebras (revised 27 February 2003) said: "... Duality operations generate isomorphism between grades k and n-k. There are several different duals, including ... the Clifford dual ... and the Hodge dual ... A dual distinct from the preceding duals is introduced by bit inversion that maps e_m -> e_mbar , where mbar is the bit inverse of m ... In particular, bit inversion transforms vectors (grade 1) ... into covectors (grade n-1) ... The bit inverse of the bit inverse is the original element ... Bit inversion does not ...[depend on the handedness of the base coordinate system]... Complementarity between space-time and momentum-energy is achieved by bit inversion, which interconverts between position representation and momentum representation. Treating momentum as a Clifford covector has the virtue of automatically enforcing the Heisenberg commutation relation as a consequence of the commutation and anti-commutation properties of the Clifford elements. ...".

Emergent Quantum

Frank Dodd (Tony) Smith, Jr. - 2011 -
- In these comments, SO(n) includes its cover Spin(n)
and x denotes various products (tensor, twisted, etc.) determined by context
and + denotes various sums (semi-direct, etc.) determined by context.

David Finkelstein and Ernesto Rodriguez in Physica D 18 (1986) 197-208 "Algebras and Manifolds ..." said: "... A sequence of generalized manifolds and Clifford algebras describes the structure of the world ... a Minkowsian simplicial manifold ... is a classical "condensation" of a quantum manifold ... described by the free Clifford algebra S of quantum set theory. ...
Let 2\r stand for a tower of r 2's.

```
              2
             .
            .
           .
2\r := 2
```

...
2\r = 1, 2, 4, 16, 64K [64K = 65,536], ... for r = 0, 1, 2, 3, 4, ...
Then S[r] has dimension N = N(r) = 2\r ...".

David Finkelstein with James Baugh, Andrei Galiautdinov, and Heinrich Saller in Journal of Mathematical Physics 42 (2001) 1489 "Clifford Algebra as Quantum Language" said: "... The Clifford algebra .. 2^(N8) ... of an octadic ... 8-dimensional ... chronon algebra ... factors [by 8-Periodicity of Real Clifford Algebras] as ... 2^(N8) = 2^8 x ... x 2^8 (N [tensor product] terms) ...".

Combining the two ideas of:
Tower-of-2 Clifford algebras Cl(0), Cl(1), Cl(2), Cl(4), Cl(16) ...
of dimensions 1, 2, 4, 16, 65,536 ...
and
Clifford Algebra 8-Periodicity tensor factors of the form Cl(8), Cl(2x8), Cl(3x8) ...
identifies

Cl(16) as the minimal basic factor building block of Real Clifford Algebras.

Cl(16) Fundamental Quantum Structure of Nested Real Clifford Algebras:

Start with Empty Set = 0

1 = Cl(0)
\
1 + 1 = Cl(1) = Cl(Cl(0))
\
1 + 2 + 1 = Cl(2) = Cl(Cl(1)) = Cl(Cl(Cl(0)))
\
1 + 4 + 6 + 4 + 1 = Cl(4) = Cl(Cl(2)) = Cl(Cl(Cl(Cl(0))))
\
1 + 16 + 120 + ... = Cl(16) = Cl(Cl(4)) = Cl(Cl(Cl(Cl(Cl(0)))))
\
1 + 65,536 + ... = Cl(65,536) = Cl(Cl(16)) =
(by Real Clifford Algebra 8-Periodicity) = Cl(16) x...(16 times)...x Cl(16)

John von Neumann said (see "Why John von Neumann did not Like the Hilbert Space Formalism of Quantum Mechanics (and What he Liked Instead)" by Miklos Redei in Studies in the History and Philosophy of Modern Physics 27 (1996) 493-510):

"... if we wish to generalize the lattice of all linear closed subspaces
from a Euclidean space to infinitely many dimensions,
then one does not obtain Hilbert space ... our "case I_infinity" ...
but that configuration, which Murray and I called "case II1" ...".

Completion of the Union of All Finite Tensor Products of Cl(16) with itself
gives a generalized Hyperfinite II1 von Neumann Factor
that in turn gives a realistic Algebraic Quantum Field Theory (AQFT).

Since Cl(16) is the Fundamental Building Block of a realistic AQFT with the
structure of a generalized Hyperfinite II1 von Neumann Factor,

**in order to understand how realistic AQFT works in detail,
we must understand the Geometric Structure of Cl(16).**

Cl(16) has 2^16 = 65,536 elements with graded structure

1
16
120
560
1820
4368
8008
11440
12870
11440
8008
4368
1820
560
120
16
1

The 16-dim grade-1 Vectors of Cl(16) are D8 = Spin(16) Vectors
that are acted upon by the 120-dim grade-2 Bivectors of Cl(16)
which form the D8 = Spin(16) Lie algebra.

Cl(16) has, in addition to its 16-dim D8 Vector and 120-dim D8 Bivector bosonic commutator structure, a **fermionic anticommutator structure** related to its sqrt(65,536) = 256-dim spinors
which reduce to **128-dim D8 +half-spinors** plus 128-dim D8 -half-spinors.

Pierre Ramond in hep-th/0112261 said:
"... the coset F4 / SO(9) ... is the sixteen-dimensional Cayley projective plane ... [represented by]... the SO(9) spinor operators [which] satisfy Bose-like commutation relations ... Curiously,
if ...[the scalar and spinor 16 of F4 are both]... anticommuting,
the F4 algebra is still satisfied ...".
The same reasoning applies to other exceptional groups that
have octonionic structure and spinor component parts, including:
E6 = D5 + U(1) + 32-dim full spinor of D5
and
E8 = D8 + 128-dim half-spinor of D8.

The Maximal Lie Algebra Geometric Structure of Cl(16) is

248-dim E8 = 120-dim D8 Spin(16) + 128-dim D8 half-spinor

E8 has a Thomas Larsson 7-grading 8 + 28 + 56 + 64 + 56 + 28 + 8
with

odd part 8 + 56 + 56 + 8
128-dim D8 half-spinor = 64s + 64c
because
E8 / D8 = E8 / Spin(16) is a rank 8 symmetric space space of Type EVIII
with 248 - 120 = 128-dimensions, the octoctonionic projective plane (OxO)P2
that corresponds to a Cl(16) half-spinor representation of Spin(16) = D8

even part 28 + 64 + 28
120-dim D8 = 28-dim D4 + 64-dim (A7+1) + 28-dim D4
because
D8 / D4 x D4 = SO(16) / SO(8) x SO(8) is a rank 8 space of Type BDI
with 64 dimensions that corresponds to A7+1 = 64v
and
D8 / (A7+1) is a rank 4 symmetric space of Type DIII
with 28 + 28 = 56 dimensions
corresponding to the Type II(8) Bounded Complex Domain
whose 36-real-dimensional Shilov Boundary
is the symmetric unitary matrices of order 8

So:

E8 = 64s + 28 + (A7+1) + 28 + 64c =

= A7 + (64s + 28 + 1 + 28 + 64c)

is the Maximal Simple Lie Algebra Structure of Cl(16)

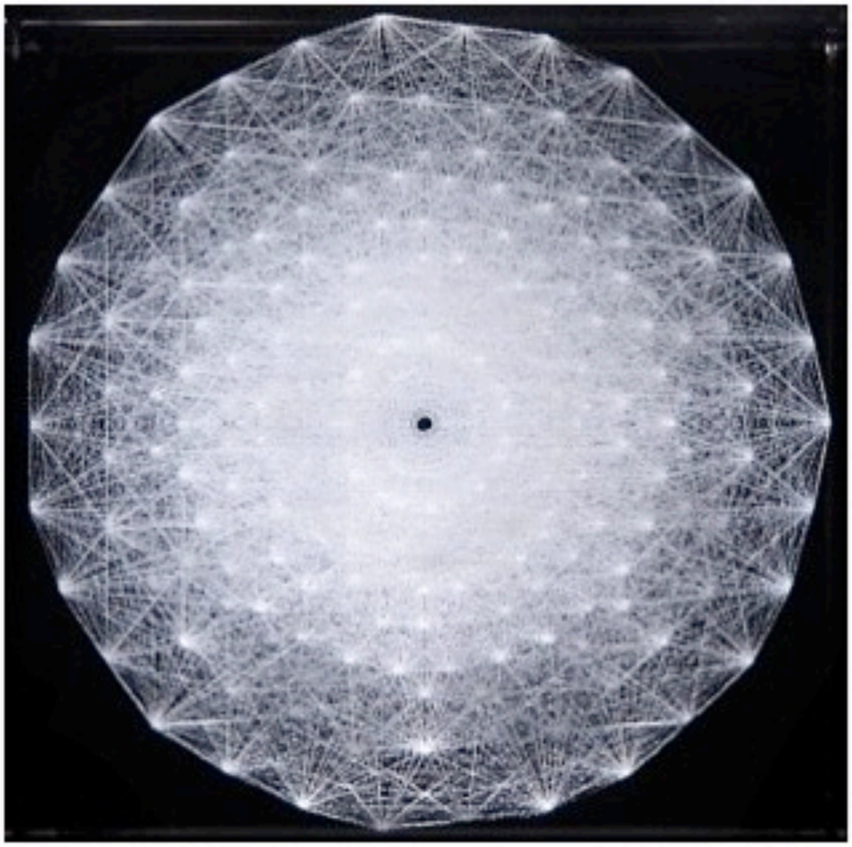
E8 Structure of Cl(16) determines a Lagrangian:

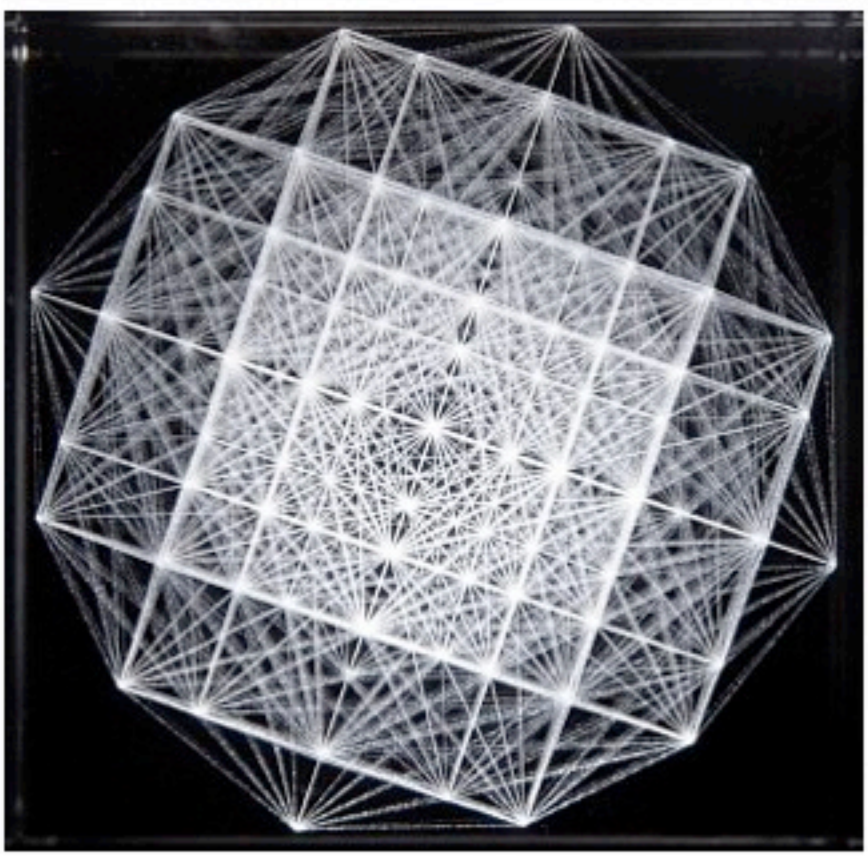
28 + 28 determine Gauge Boson Terms
for Gravity and the Standard Model

64s + 64c determine Dirac Fermion Terms
for
8 Components of 8 Particles
and
8 Components of 8 Antiparticles

A7+1 = Gl(8) determines 8-dim Spacetime 64v
and
A7 = Sl(8) determines 8-dim Volume
for
8-dim Spacetime Base Manifold
with 7 distinct E8 Lattice Structures.
Selecting One Particular E8 Lattice
effectively
Freezes Out a Quaternionic Subspace of E8 Spacetime
producing:
8-dim Kaluza-Klein M4xCP2
Second and Third Generation Fermions
Batakis Standard Model Gauge Groups
Mayer-Mechanism Higgs
3-Level Higgs-Tquark Condensate System

Here are two views of the 240 Root Vectors of E8:
(images from Bathsheba Grossman glass)





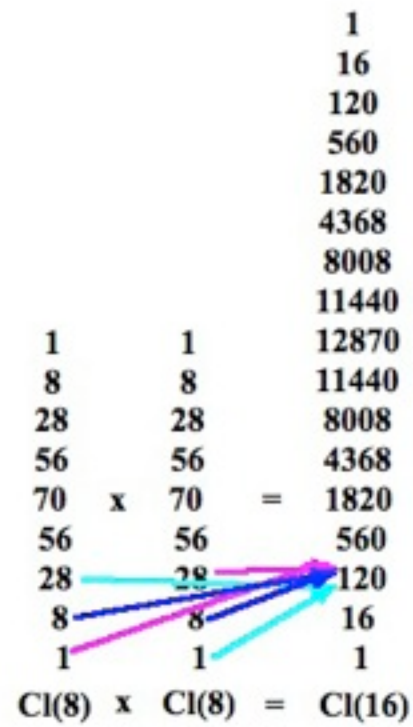
The Real Clifford Algebra 8-Periodicity factoring

				1
				16
				120
				560
				1820
				4368
				8008
				11440
1		1		12870
8		8		11440
28		28		8008
56		56		4368
70	x	70	=	1820
56		56		560
28		28		120
8		8		16
1		1		1

Cl(8) x Cl(8) = Cl(16)

gives the 16-dim D8 Vector of Cl(16) as 16 = 1x8 + 8x1
which is the sum of **two copies of the 8-dim D4 Vector of Cl(8) = RP1 x S7**
that is the 8-real dimensional Shilov Boundary
of the Type IV(8) Bounded Complex Domain
corresponding to the rank 2 Symmetric Space of Type BDI SO(10) / SO(8) x U(1)
which has 45-28-1 = 16 real dimensions (8 complex dimensions).

The Real Clifford Algebra 8-Periodicity factoring



gives the 120-dim Cl(16) grade-2 Bivectors as the sum of these tensor products:
Cl(8) grade-0 x Cl(8) grade-2 = 1 x 28 = 28
Cl(8) grade-1 x Cl(8) grade-1 = 8v x 8v = 64v
Cl(8) grade-2 x Cl(8) grade-0 = 28 x 1 = 28
so that **120-dim Cl(16) grade-2 Bivectors = 28 + 64v + 28**

The 120-dim D8 Spin(16) Bivector of Cl(16) is symmetry of the rank 2 Type BDI Symmetric Space SO(18) / SO(16) x U(1)
which has 153-120-1 = 32 real dimensions (16 complex dimensions)
and corresponds to the Type IV(16) 16-complex dim Bounded Complex Domain
with 16-real dimensional Shilov Boundary RP1 x S15
RP1 x S15 = RP1 x S7 x S8 (due to Hopf Fibration S7 -> S15 -> S8)
RP1 x S15 = RP1 x S3 x S4 x S8 (due to Hopf Fibration S3 -> S7 -> S4)

The Real Clifford Algebra 8-Periodicity factoring, in terms of spinors,

Cl(8) x Cl(8) = Cl(16)

(16 = 8s+8c) x (16 = 8s+8c) = 256 = (8sx8s + 8cx8c) + (8sx8c + 8cx8s)
(8sx8s + 8cx8c) = 128 = 64s + 64c

gives 256-dim Cl(16) full spinors as the sum of these tensor products:
Cl(8) half-spinor 8s x Cl(8) half-spinor 8s
Cl(8) half-spinor 8s x Cl(8) half-spinor 8c
Cl(8) half-spinor 8c x Cl(8) half-spinor 8s
Cl(8) half-spinor 8c x Cl(8) half-spinor 8c
so the

128-dim Cl(16) half-spinor = 8s x 8s + 8c x 8c = 64s + 64c

E8 has triality transformations among 64v and 64s and 64c consistently inherited from Cl(8) Spin(8) D4 triality among 8v and 8s and 8c

By Contraction of E8
Cl(16) has a Maximal Nilpotent Heisenberg Algebra Structure:

David Finkelstein in a 2003 ZKM Karlsruhe presentation at
https://www.physics.gatech.edu/files/u9/publications/0006.pdf
said: "... We model the cosmos ... using a high-order Clifford algebra ...
algebra (or group) expansion ...[is]... a process that Segal discovered ...
To expand a Lie algebra or its group is to insert a small parameter p called the expansion parameter into the algebra multiplication table so that the algebra changes beyond isomorphism, no matter how small the parameter. ...
The expanded theory reduces to the unexpanded theory when the constant p goes to 0, **the process called contraction**. ...".

Rutwig Campoamor-Stursberg in Acta Physica Polonica B 41 (2010) 53-77 said:
"... We have classified all **contractions** of complex simple exceptional Lie algebras onto
semidirect products ... s + h_N ... of semisimple and Heisenberg algebras.
An analogous procedure holds for the real forms of the exceptional algebras ...
Contractions of E8 ... E8 contains D8 contains A7 ...[and for E8]... N = 92
... This reduction gives rise to the contraction ..."

248-dim E8 -> A7 + h_92 =
= A7 + (64s + 28 + 1 + 28 + 64c)

where
the Heisenberg algebra h_92 is made up of the central **1**
plus
a gauge boson commutator part 28 + 28
and
a fermion anti-commutator part 64s + 64c

Nilpotent Heisenberg Algebras:

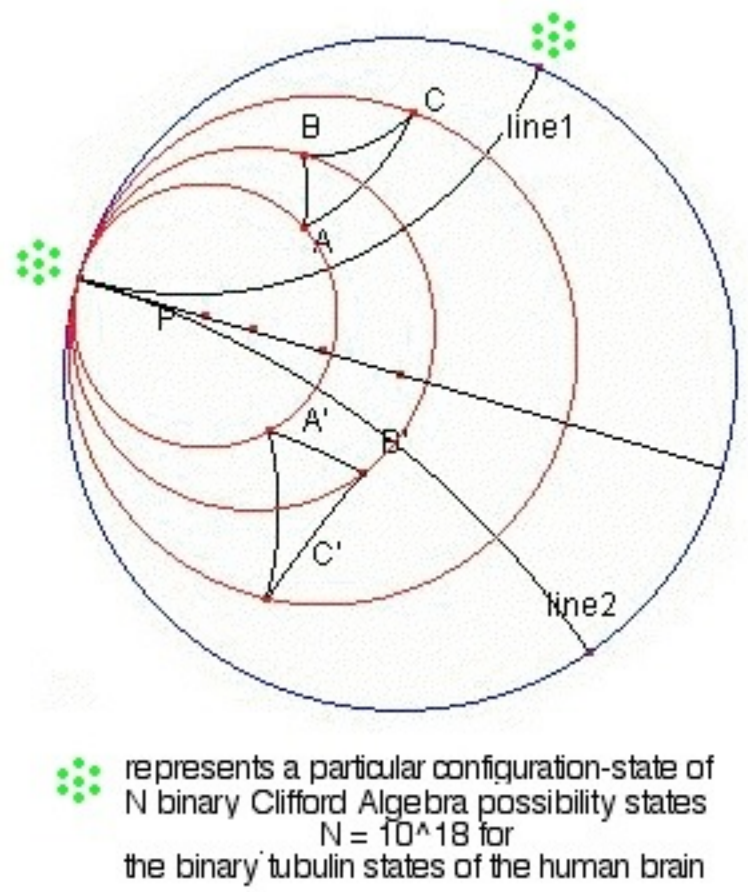
Consider the h_92 Heisenberg part of H248 = SU(8) + h_92.

h_92 is a 92 + 1 + 92 = 185-dim algebra that is nilpotent
and describes
creation operators for 8px8cp = 64 fermion particle component states
and 8px8cp = 64 fermion antiparticle component states
(equivalent to creation and annihilation operators for fermion particles)
and
creation operators for 28 Gravity bosons and 28 Standard Model bosons
(equivalent, since bosons are their own antiparticles,
to creation operators for 16 Gravity + 12 Standard Model bosons
and annihilation operators for 16 Gravity + 12 Standard Model bosons)

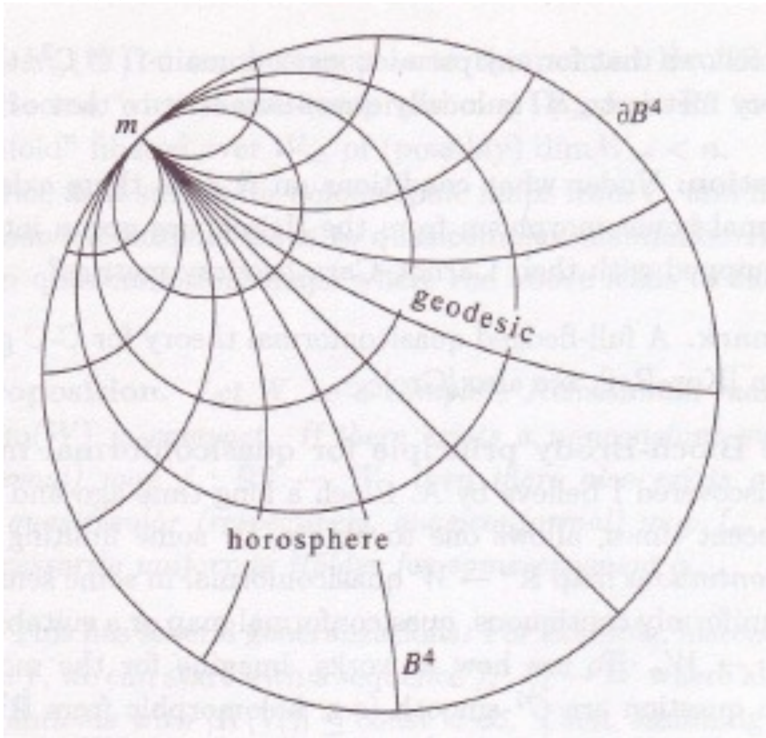
A naive construction of Hilbert space from creation and annihilation operators does not give a realistic quantum theory, but a generalized II1 hyperfinite II1 von Neumann factor algebra does give a realistic Algebraic Quantum Field Theory.
Miklos Redei in Studies in the History and Philosophy of Modern Physics 27 (1996) 493-510 said: "... John von Neumann ... [said that]... Hilbert space vectors ... represent the physical states ... redundantly ...
if we wish to gneralize the lattice of all linear closed subspaces from a Euclidean space to infinitely many dimensions,
then one does not obtain Hilbert space ... case Ioo,
but
... case II1 ... a type II1 (factor) von Neumann algebra ...".

Heisenberg algebras are useful in describing the Quantum Physics of the Interior of the physical spacetime Lie Ball B4.

If properly understood, that physics might describe Quantum Consciouness Resonant Connections between,
for example,
two human brains located at different points
on the M4 spacetime Lie Sphere Shilov Boundary of the B4 Lie Ball:



Misha Gromov in his book "Metric Structures for Riemannian and Non-Riemannian Spaces" (Birkhauser 2001), and Stephen Semmes in Appendix B, said: "... Let B4 be equipped with its Bergman metric ... so that B4 is isometric to the complex hyperbolic plane,
H_C^2 = (U(1) x U(2)) \ U(1,2)



... ...

The action of U(1,2) also preserves [the boundary] bB4 ... and the field of 2-planes on bB4 .... the stabilizer of a point of bB4 contains a subgroup isomorphic to the Heisenberg group ...".

R. Coquereaux says, in his paper " Lie Balls and Relativistic Quantum Fields", Nuc. Phys. B. 18B (1990) 48-52: "... In the present paper, we are mainly interested in **the four dimensional (complex) Lie ball [B4] that we shall denote by D**. This smooth manifold can be written as **SOo(4,2) / SO(4)xSO(2)** or as SU(2,2) / S(U(2)xU(2)). ... D is a bounded non compact symmetric domain of type I and IV. ... The metric of D is euclidean and blows up near the boundary (as in the usual geometry of Lobachevski) but ... induces a conformal Lorentz structure on the boundary. The domain D is a Lie ball ... the Shilov boundary (compactified Minkowski Space-Time) can be defined as the Lie sphere ... The domain D also admits an unbounded realization: the future tube. ... This ... unbounded realization of the Lie ball admits a simple physical interpretation ... the imaginary part y of z = x + iy can be interpreted as the inverse of a momentum ... Points of the domain D describe therefore both the position (in space and time) and the momentum (with

$p^2 > 0$) associated with a physical event. **The domain itself becomes therefore a curved relativistic phase-space**. ... The group SO(4,2) ... as the group of conformal transformations for a Lorentzian Space-Time of signature (1,3) coincides with the group of analytical diffeomorphisms of the Lie ball D ...
[Using] the physical interpretation of Space-Time with the Shilov boundary ... S ... of D together with the interpretation of the imaginary part of the complex variable ... as the inverse of a momentum ...
Physics is "simple" (and euclidean) in the domain D ...
many of the difficulties of classical or quantum physics arise because
we try to go to the "boundary" and to formulate the laws of Physics there
...
what we have here is rather a (Radon)-Gelfand-Graev transformation -
i.e. integration over horocycles ...
its physical interpretation is quite different ...[from]... the Fourier transformation ...".

Bohm and Hiley say, in their book "The Undivided Universe" (Routledge 1993, section 15.9): "... the theory of the higher spherical geometry of Lie and Klein ... relate[s to]... light cones. ... Each light cone intersects ...[a]... hyperplane ... say at $t = 0$ ... in a sphere. Those light cones whose vertices have $t < 0$ will be defined to produce spheres of positive radius $r = t$. The light cones whose vertices have $t > 0$ will also produce spheres but these will be assigned negative radii.
So there is a 1 : 1 correspondence between light cones and their intersections with any hyperplane t = const. ... Lie and Klein proposed ... an interesting parameterization of these spheres ...

$$X_6^2 - X_5^2 - X_4^2 + X_1^2 + X_2^2 + X_3^2 = 0$$

[where]... the coordinates of the center are (a,b,c) and the radius is r ... introduce homogeneous coordinates by writing ...[

$$r^2 - a^2 - b^2 - c^2 = (X_6 - X_5) / (X_6 + X_5)$$
$$X_1 = (X_6 + X_5)\, a$$
$$X_2 = (X_6 + X_5)\, b$$
$$X_3 = (X_6 + X_5)\, c$$
$$- X_4 = (X_6 + X_5)\, r$$

]... The advantage of this parameterization is that the conformal invariance of the description is now evident ...
[it]... implies the invariance to six dimensional pseudo-orthogonal transformations which are ... isomorphic to the conformal group. ...
the condition for two spheres to come into contact ... is

$$X_6 Y_6 - X_5 Y_5 - X_4 Y_4 + X_1 Y_1 + X_2 Y_2 + X_3 Y_3 = 0$$

where S and Y refer respectively to the spheres in question ...
if two spheres are in contact, then the vertices of their corresponding light cones are connected by a null ray ...

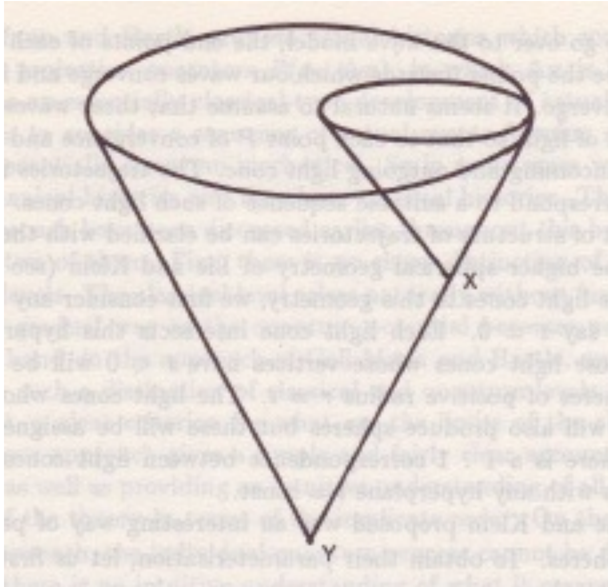


Figure 15.4: Light cones on one light ray yx

... Let us now consider our trajectory starting from a sphere of zero radius representing a light cone with vertex at $t = 0$. The first step in our trajectory ...

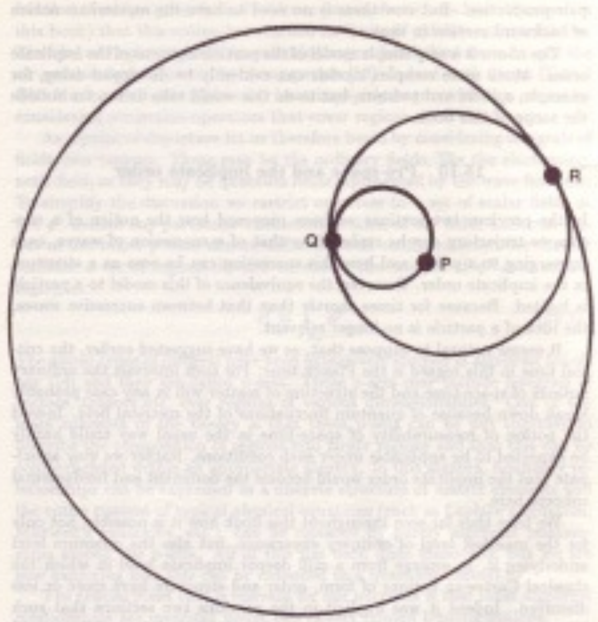


Figure 15.7: Trajectory in terms of spheres

... is represented by the point of contact P and a sphere of radius r.

The corresponding null ray is in the direction of the radius of the sphere at the point P.
The next null ray will be represented by a larger sphere contacting the first sphere at the point Q.
The next null ray will correspond to a still larger sphere contacting the second sphere at another point R.
This procedure is to be continued indefinitely so that we obtain a complete description of the zigzag trajectory. ... we have described our trajectory entirely in terms of wave forms at any given time. ... this description is conformally invariant. ... in a time displacement, dt, the radius of each sphere changes by dt. ... in a Lorentz transformation, however, the radii and centers change together so that initially concentric spheres cease to be concentric. ... our trajectory is now defined in an implicate order. As this order unfolds in time, the radii of the spheres corresponding to incoming waves decrease while those corresonding to outgoing waves increase. The entire trajectory is now shown as a kind of enfolded geometric structure ... The law of development is implicit in this structure. ... Feynman describes his trajectories as a sequence of points in time. We describe them as sequences of spheres all enfolded at a given time. The Lagrangian is thus, in our approach, a property of the implicate order that holds at any given moment ... backwards tracks in time are replaced by tracks in which the implication parameter is decreasing. In our model, the implication parameter is just the radius of the sphere. Thus ...

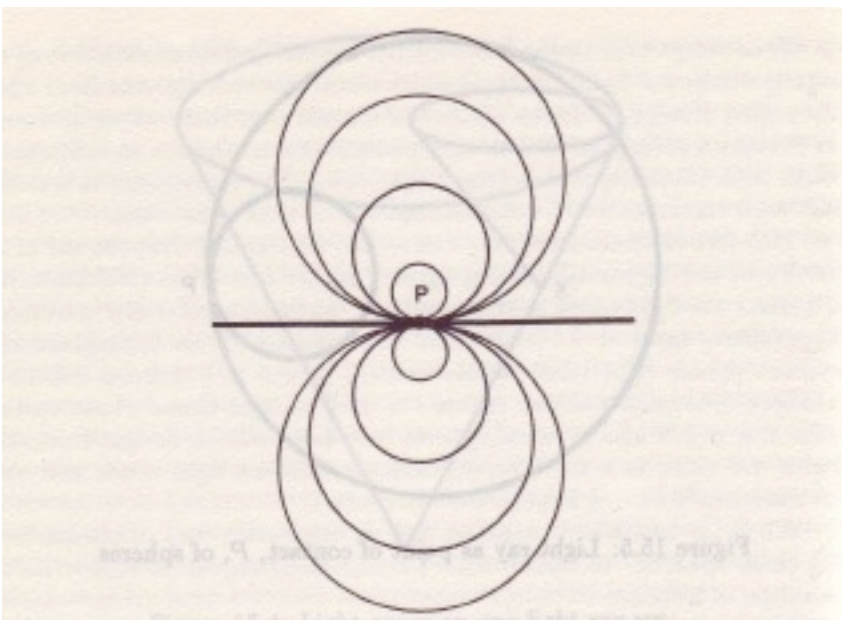


Figure 15.6: Illustration of spheres with positive and negative radii

... we could have the radii of spheres decrease for a while and then increase again. ... this corresponds to pair production. ...".

In his review (Bull. A.M.S. 32 (1995) 441-446)) of Helgason's book Geometric Analysis of Symmetric Spaces (AMS 1994), Francois Rouviere asks a question:
"...What are ... natural substitutes for the Euclidean lines in the hyperbolic unit disk X = H2 = SU(1,1) / SO(2) ?
first answer:
take as Z the set of all geodesics of X (circles orthogonal to the unit circle). The corresponding Radon transform may be called a generalized X-ray transform, recalling ... the mathematical theory of tomography ...
A second answer is:
take Z as the set of all horocycles of X, that is, the "wave surfaces" orthogonal to a "parallel beam of rays" (geodesics meeting at infinity on the unit circle).
The horocycles are thus all circles inwardly tangent to the unit circle.
Both settings are considered in the book ...
Helgason deals with the following Radon transforms:
integration over k-dimensional totally geodesic submanifolds in a space with constant curvature: Rn, Hn, Sn ("geodesic transform");
Radon transform for Grassmannians,
with a specific incidence relation between p-planes and q-planes in R(p+q+1);
Poisson integrals for bounded domains:
for example, if G = SU(1,1) acting on the complex plane and if K and H are the isotropy subgroups of the points 1 and 0 respectively, then X is the unit circle, Z is the unit disk, and Rf is the classical Poisson integral of f;
integration over horocycles in a Riemannian symmetric space
of the noncompact type ("horocycle transform") ...
...
when dim X < dim Z and the range of R can be characterized as the kernel of a certain differential operator on Z ...
This extends the Poisson integral example ... where X is the circle, Z is the disk, and the range of R is known to be the kernel of the Laplace operator on the disk. ...
The Fourier transform ... on a Riemannian symmetric space of the noncompact type X = G / K ... is ...

f^(l,b) = INTEGRAL(X) f(x) exp(- i l + r , A(x,b) dx .

It is inverted by

f(x) = INTEGRAL(a*xB) f(x) exp(- i l + r , A(x,b)) | c(l) |^(-2) dl db .

... as a function of x,
the exponential is an eigenfunction of all G-invariant differential operators on X ...
Horocycles are ... level surfaces of A(x,b) for fixed b ...
the Plancherel measure | c(l) |^(-2) dl db involves Harish-Chandra's celebrated, and explicitly known, c-function. ...
the Poisson kernel for the unit disk is an exponential of the function A(x,b) ...".

Harald Upmeier says in his paper "Weyl Quantization of Complex Domains":
"... Every ... bounded symmetric domain ... D ...
has a Harish-Chandra realization as the open unit ball
of a complex vector space V = Cn, with respect to some norm ...
The invariant differential operators on D
(called "higher Laplacians" in the physics literature) ...
form a polynomial algebra in r generators d1, d2, ..., dr
[of]... order 2, 4, ..., 2r, respectively
(d1 is the ordinary non-euclidean Laplacian on D) ...
the Lie balls of rank 2 .. have immediate interpretations in physical terms,
since D is related to I. Segal's conformal universe
and also to the forward light cone ...".

Weyl Symmetric Polynomial Degrees and Topological Types of some groups related to E8 Physics and H248:

E8: degrees - 2, 8, 12, 14, 18, 20, 24, 30;
note that 1, 7, 11, 13, 17, 19, 23, and 29 are all relatively prime to 30.
type - 3, 15, 23, 27, 35, 39, 47, 59; center = Z1 = 1 = trivial

D8 Spin(16): degrees - 2, 4, 6, 8, 10, 12, 14, 8
type - 3, 7, 11, 15, 19, 23, 27, 15; center = Z2 + Z2

E6: degrees - 2, 5, 6, 8, 9, 12
type - 3, 9, 11, 15, 17, 23; center = Z3

D5 Spin(10): degrees - 2, 4, 6, 8, 5
type - 3, 7, 11, 15, 9; center = Z4

D4 Spin(8): degrees - 2, 4, 6, 4
type - 3, 7, 11, 7; center = Z2 + Z2

D3 Spin(6) = A3 SU(4): degrees - 2, 4, 3
type - 3, 7, 5; center = Z4

E7: degrees - 2, 6, 8, 10, 12, 14, 18
type - 3, 11, 15, 19, 23, 27, 35; center = Z2

D6 Spin(12): degrees - 2, 4, 6, 8, 10, 6
type - 3, 7, 11, 15, 19, 11; center = Z2 + Z2

F4: degrees - 2, 6, 8, 12; note that 1, 5, 7, and 11 are all relatively prime to 12.
type - 3, 11, 15, 23; center = Z1 = 1 = trivial

B4 Spin(9): degrees - 2, 4, 6, 8
type - 3, 7, 11, 15; center = Z2

G2: degrees - 2, 6; note that 1 and 5 are both relatively prime to 6.
type - 3, 11; center = Z1 = 1 = trivial

A2 SU(3): degrees - 2, 3
type - 3, 5; center = Z3

Some Symmetric Spaces related to E8 Physics and H248:

Type EVIII rank 8 symmetric space E8 / D8 is not Hermitian
is 128-dim octooctonionic projective plane (OxO)P2

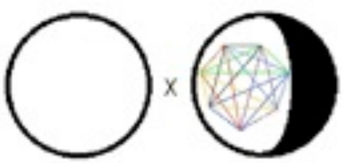
Type EIX rank 4 symmetric space E8 / E7xSU(2) is not Hermitian
is 112-dim root vectors of D8 set of (QxO)P2 in (OxO)P2

Type EVI rank 2 symmetric space E7 / D6xSU(2) is not Hermitian
is (QxO)P2

Type EVII rank 3 symmetric space E7 / E6xU(1) is Hermitian Type VI
is 54-dim set of (CxO)P2 in (QxO)P2

Type EIII rank 2 symmetric space E6 / D5xU(1) is Hermitian Type V
is 32-dim projective plane (CxO)P2
The Shilov boundary S of the corresponding Bounded Complex Domain
is not tube type and "... has CR-dimension 8 and CR-codimension 8. On S the group Spin(10) acts transitively and the reduction ... of S is the symmetric Hermitian manifold SO(10)/(SO(2)xSO(8)) ..." (from math/9905183 by Kaup and Zaitsev)

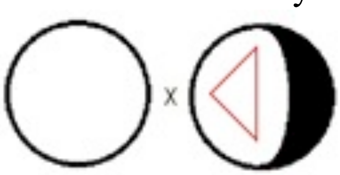
Type BDI rank 2 symmetric space D(5) / D4xU(1) is Hermitian Type IV(4)
is 16-real-dim real 8-Grassman manifold of R18
The Shilov boundary of the corresponding Bounded Complex Domain



is the Lie Sphere RP1 x S7

Type BDI rank 2 symmetric space D(4) / D3xU(1) is Hermitian Type IV(3)
is 12-dim real 6-Grassman manifold of R14

Type BDI rank 2 symmetric space D(3) / D2xU(1) is Hermitian Type IV(2)
is 8-real-dim real 4-Grassman manifold of R10
The Shilov boundary of the corresponding Bounded Complex Domain



is the Lie Sphere RP1 x S3

Zero-Divisors and Sedenions:

The h_92 part of H248 is a Nilpotent 185-dimensional Heisenberg Lie algebra for 92 sets of creation-annihilation operators:

8px8cp = 64 Fermion Particle Creators + 8px8cp 64 Fermion AntiParticle Creators

28 Gravity Boson Creators + 28 Standard Model Boson Creators

plus

1 central Heisenberg Algebra element

8p of the 8px8cp correspond to the 8 fundamental fermion particles.
8p of the 8px8cp correspond to the 8 fundamental fermion antiparticles.

Since each of the 8p particles annihilate the corresponding 8p antiparticles, the 16-dim Sedenion algebra of 8p + 8p has an 8p+8p = 16-dim space of Zero-Divisor with graded structure

```
      1 + 3 + 3 + 1
+         1 + 3 + 3 + 1
=     1 + 4 + 6 + 4 + 1
```

of the Sedenion Zero-Divisors 1 + G2 + 1

The relationship between Nilpotent Heisenberg algebras and Zero Divisors has been noted by J. M. Landsberg and L. Manivel in math/0402157
where they said:
"... the sextonions ... a six-dimensional alternative algebra, with zero divisors ...
sit between the quaternions and octonions to produce
a Lie algebra sitting between E7 and E8 ...[that is E7 + h_28
where +]... denotes the semi-direct product ...
the Heisenberg algebra [h_28] is the radical. ...".

Zero Divisors, Graded Structures, and Associative Triples

There are multiple ways in which the (1+14+1) = 16-dimensional Zero Divisor space of the Sedenions represented by R+G2+R with graded structure
1 + **4** + **6** + **4** + **1** can be deleted from the Cl(8) grading
1 + 8 + **28** + 56 + **70** + 56 + **28** + 8 + **1**
leaving the graded structure 8 + 24 + 56 + 64 + 56 + 24 + 8
of the 240 Root Vectors of the Lie Algebra E8.

There is only one way that the scalar 1+1 of R+G2+R can be deleted from the scalar 1+1 of Cl(8).

Just as the 28-dimensional D4 Lie algebra has 7 independent sets of 4 Cartan subalgebra generators, there are 7 different ways that the first 4 of the 1+4+6+4+1 Zero Divisors can be deleted from the first 28 of the 1+8+28+56+70+56+28+8+1 of Cl(8).

Choosing which of the 7 ways for the first 4 and 28 fixes, by dualities in the graded structures, the choices for the second 4 to be deleted from the second 28 and for the middle 6 to be deleted from the middle 70, so there are **7 ways that R+G2+R can be deleted from Cl(8)**.

Those 7 ways correspond to the 7 different independent E8 lattices in 8-dimensional Euclidean space, each of which has its own 240-vertex Witting Polytope configuration as the first shell around the origin, and therefore its own set of 240 Root Vectors.

The also correspond to the 7 Associative Triples of the Octonions.

Quaternions have only one Associative Triple { i , j , k }
For the octonions,
6 new associative triple cycles appear
{ i , J , K }
{ I , j , K }
{ I , J , k }
{ i , E , I }
{ j , E , J }
{ k , E , K }

They correspond to the Lie algebra Spin(4).
The other 35 - 7 = 28 triples are not cycles.

Denote the 7+8 = 15 sedenion Imaginary basis elements
by { i, j, k, E, I, J, K, S, T, U, V, W, X, Y, Z } .

The sedenions correspond to a tetrahedron,
a 3-dimensional simplex,
4 vertices v of the tetrahedron corresponding to EIJK ;
6 edges e of the tetrahedron corresponding to ijkTUV ;
4 faces f of the tetrahedron corresponding to WXYZ ;
and the 1 entire tetrahedron T corresponding to S .
There are 4+6+4+1 = 15 things.
There are 35 (projective) lines each with 3 things and
they correspond to the 35 associative triples of the
sedenions.
Geometrically, they are of the form:
3+4 = 7 corresponding to the 7 associative triples of
octonions:
3 like eTe (where e is opposite e on the whole tetrahedron
T);
4 like vTf (where v is opposite f on T);
and
16+12 = 28 corresponding to 28 new ones formed at the
sedenions:
4 like eee (where eee are all on the same face);
6 like vev (these are the edges);
6 like fef (where the edge of e is not on f or f,
that is, f and f are opposite to e);
12 like vfe (where v is opposite e on face f).

The sedenion multiplication table is 16x16
so it has 256 = 2^8 entries and can be written
as a 16x16 matrix:

```
   r i j k E I J K S T U V W X Y Z
 r x x x x x x x x x x x x x x x x
-i x x
-j x   x q
-k x     x
-E x       x o o o
-I x         x o o
-J x           x o
-K x             x
-S x               x s s s s s s s
-T x                 x s s s s s s
-U x                   x s s s s s
-V x                     x s s s s
-W x                       x s s s
-X x                         x s s
-Y x                           x s
-Z x                             x
```

The 16+15+15 = 46 x entries denote the "real" products that
cannot belong to an associative triple cycle of the type
ijk.

For the ri part of the table, the complex numbers,
there are no associative triple cycles.

For the rijk part of the table, the quaternions,
there is only one associative triple cycle,
the ijk triple itself, denoted by the q entry.

The 6 o entries represent the 6 new
associative triple cycles that come with the octonions.

The 28 s entries represent the 28 new
associative triple cycles that come with the sedenions.

The 28 new associative triple cycles of the sedenions are related to the 28-dimensional Lie algebra Spin(0,8), and to the 28 different differentiable structures on the 7-sphere S7 that are used to construct exotic structures on differentiable manifolds.

WHAT ABOUT GOING UP TO HIGHER DIMENSIONS?

For 32-ons, we get 120 new associative triple cycles, and they represent the Lie algebra Spin(0,16) of the Clifford algebra Cl(0,16).

HOWEVER, NOTHING REALLY NEW HAPPENS BECAUSE OF THE PERIODICITY PROPERTY OF REAL CLIFFORD ALGEBRAS.

The periodicity theorem says that
Cl(0,N+8) = Cl(0,8) x Cl(0,N)   (here x = tensor product)

That means that the Clifford algebra Cl(0,16) of the 32-ons is just the tensor product of two copies of the Clifford algebra Cl(0,8).

So, everything that happens in the 32-on Clifford algebra is just a product of what happens with Cl(0,8).

That point is emphasized by the fact (see Lohmus, Paal, and Sorgsepp, Nonassociative Algebras in Physics (Hadronic Press 1994)) that the derivation algebra of ALL Cayley-Dickson algebras at the level of octonions or larger, that is, of dimension 2^N where N = 3 or greater, is the exceptional Lie algebra G2, the Lie algebra of the automorphism group of the octonions.

The exceptional Lie algebra G2 is 14-dimensional, larger than the 8-dimensional octonions, but smaller than the 16-dimensional sedenions.

The Associative Triples are discussed by Guillermo Moreno (who uses the term "special triple" for them) in math/0512516

"... For $n \geq 4$, Eakin-Sathaye showed that Aut(A_n) = Aut(A_(n-1)) x S_3
Where S_3 is the symmetric group of order 6 ...
We will describe the set
M(A_m , A_n) = { F : A_m -> A_n | F algebra monomorphism } ...
[special triple = associative triple = associative triangle]...
For a special triple {a,b,c} in A_n and $n \geq 3$ ...
f(a,b,c) = Span{ e_0 , a , b , ab , c(ab) , cb , ac, c } ...
is an eight-dimensional vecor subspace isomorphic, as algebra,
to A_3 = f the octonions and
M(A_3 , A_n) = { (a,b,c) in (A_n)^3 | {a,b,c} special triple } ...
Suppose that $n \geq 4$ and that {a,b,c} is a special triple in A_n ... any orthonormal triple {x,y,z} of pure elements in f(a;b;c) with z perpendicular to (xy) is also a special triple in A_n ...
The main result of this paper is ...[that]... the set of type II monomprphsims from A_3 to A_(n+1) can be described by the set of zero divisors in A_(n+1) for $n \geq 4$ ... this set is ... complicated to describe ...".

Guillermo Moreno had written some background in an earlier paper "The higher dimensional Cayley-Dickson algebras" that he sent to me around the summer of 2000 for which I have no publication reference
"... the group of automorphisms ...[of]... the Cayley-Dickson algebras, denoted by A_n = R^(2^n) ... for $n \geq 4$ ... is isomorphic to G2 x F_n where F_n is a finite group, in fact F_n is the product of (n - 3) copies of S_3 the symmetric group of order 6 (see ... Eakin-Sathaye, On automorphisms and derivations of Cayley-Dickson algebras, Jornal of Pure and Applied Algebra, 129, 263-278 (1990) ...) ...
question ... the real Stiefel manifold ... V_(2^n - 1 , 2) consists ... of zero divisors an A_(n+1) for $n > 3$? ...
Any zero divisor in A_n is double pure ...
The zero divisors in A_n , $n \geq 4$ form real algebraic variety in ... R^(2^n - 2) ...
The set of non-zero divisors in A_n ($n \geq 4$) is a open dense subset ...
singular elements ... are the zero divisors and regular elements are the non-zero divisors and ... the rank ... is r = 2 ...
The real algebraic variety defined by the zero divisors in A_n for $n \geq 4$ has at most (2^(n-2) - 2) irreducibles components ...
For ... $n \geq 4$ ... alpha = (a,b) ... =/= 0 ... is a zero divisor if and only if F(alpha) is a zero divisor ... for all F in O(2) ... If alpha ... is a Stiefel element ...
then F(alpha) is also Stiefel element for all F in O(2) ...".

In a later paper at math/0512517 Guillermo Moreno wrote about the "complicated to describe" zero divisors
"... For n = 4 the set of zero divisors in A_4 of fixed norm can be identified with V_(7,2) the real Stiefel Manifold of two frames in R^7 and the singular set of (x,y) = 0 and ||x|| = ||y|| = 1 is homeomorphicto G2 the exceptional simple Lie group of rank 2. ... The description of the zero divisors in A_4 is given by the known fibration G2 -pi-> V_(7,2) with fiber S3 [the 3-sphere] since all the nontrivial annihilators are 4 dimensional.
For n ≥ 5 there is NO analogous description. We will show that the zero divisors are in A_(n+1) and V_(2^n - 1 , 2) are related, but they are not equal and the corresponding singular set has (unknown) complicated description. ...
Any zero divison in A_n is double pure ... we define a suitable O(2)-action on the double pure elements of A_n ...
in contrast with the case n = 4 where the zero divisors must have coordinates in A_3 of equal norm ... this is no the case for A_5 ... Therefore the zero divisors in A_5 are "very far" to be described s in A_4 where they can be identified with V_(7,2) the Stiefel Manifold ...[SO(7) / SO(5)]...
But also ... the set of zero divisors in A_(n+1) has some subset which can be described in terms of the Stiefel Manifold V_(2^n - 1 , 2) for n ≥ 3 ...
... the set of Stiefel elements ... with entries of norm one ... can be seen as the real Stiefel manifold V_(2^n - 1 , 2)
... Is any Stiefel element ... a zero divisor ? ... Open Question ...".

M. Nakahara describes Stiefel Manifolds in "Geometry, Topology and Physics" (Adam Hilger (1990)
"... The Stiefel manifold V(m,r) is ... SO(m) / SO(m-r) ... The Stiefel manifold is, in a sense, a generalization of a sphere ... V(m,1) = S^(m-1) ... and
dim V_(m,r) = [r(r-1)]/2 + r(m-r) ...".
Therefore, dim V_(m,2) = 1 + 2(m - 2) = 2m -3 .

Note that Lie Spheres are described by the Conformal Group symmetric space

SO(m) / (SO(m-2) x SO(2)) of dimension 2m - 4 .

Consider the Stiefel Manifolds V_(N,2) = SO(N) / SO(N-2)
and Conformal Lie Spheres SO(N) / (SO(N-2) x SO(2))
and the fibrations:

```
V_(7,2)=SO(7)/SO(5) ->        G2             ->  S3=SU(2)
    dim = 11                dim = 14             dim = 3


SO(7)/(SO(5)xSO(2)) ->  V_(7,2)=SO(7)/SO(5) -> SO(2)=U(1)=S1
    dim = 10                dim = 11              dim = 1
```

The Zero Divisors of A_n consist of a number of Irreducible Components, the maximum possible number of which is 2^(n-2) - 2 .

For the Sedenions A_4 the maximum number of Irreducible Components is 2 .

Each Zero Divisor Irreducible Component for A_n is related to the Stiefel Manifold V_(2^(n-1) - 1 , 2) .

Each Stiefel Manifold SO(2^(n-1) - 1) / SO(2^(n-1) - 3)
is the fibre product S1 x SO(2^(n-1) - 1) / (SO(2^(n-1) - 3) SO(2)) =
S1 x LieBall(2^(n-1) - 1)
so the (2(2^(n-1) - 1) - 4) = (2^n - 6)-dimensional LieBall(2^(n-1) - 1)
represents the core of each Irreducible Component of the Zero Divisors of A_n
with the core enhanced by S1 = SO(2) = U(1) .

For the Sedenions A_4
the singular set of (x,y) = 0 and ||x|| = ||y|| = 1 is homeomorphic to G2 .
Adding x and y scalar dilations gives R + G2 + R .
G2 is the fibre product S3 x SO(7) / SO(5) = S3 x V_ (7 , 2) =
= S3 x S1 x SO(7) / (SO(5) x SO(2)) = S3 x S1 x LieBall(7)
so the 10-dimensional LieBall(7) represents the core of each Irreducible
Component of the Zero Divisors of the A_4 Sedenions
with the core enhanced by S3 x S1 = SU(2) x U(1) = U(2) and adding R + R .

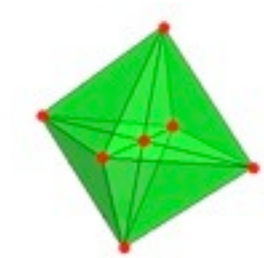
Consider the following sequence of Cayley-Dickson algebras A_n :

n	CDA	Im	AssocTriple	dim Aut(A_n)	dimLieSph	maxIrComp	MaxTotal
1	Complex	1	0	0=Z/2	0		
2	Quaternion	3	1	3=SU(2)	0		
3	Octonion	7	1+6=7	14=G2	0		
4	Sedenion	15	7+28=35	84=14x1x6	10	2	20
5	32-ons	31	35+120=155	168=14x2x6	26	6	156
6	64-ons	63	155+496=651	252=14x3x6	58	14	812
7	128-ons	127	651+2016=2667	336=14x4x6	122	30	3660
8	256-ons	255	2667+8128=10795	420=14x5x6	250	62	15500
9	512-ons	511		504	506		

Only the Quaternions, Octonions, and Sedenions have Excess Associative Triples, because only for them is the number of Associative Triples greater than the maximum number of Irreducible Zero Divison Components times the dimension of the core Lie Sphere of each Zero Divisor Component:

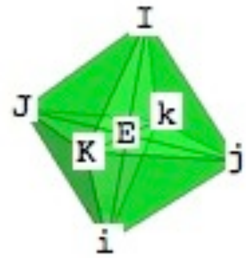
Quaternions have no Zero Divisors and only one Associative Triple.

Octonions have no Zero Divisors and 7 Associative Triples,
which correspond to the 7 Imaginary Octonion basis elements { i, j, k, E, I, J, K }
and to a 7-vertex configuration called by Arthur Young the Heptahedron (also independently developed by Onar Aam), which is composed of the 6 vertices of an Octahedron plus a central point

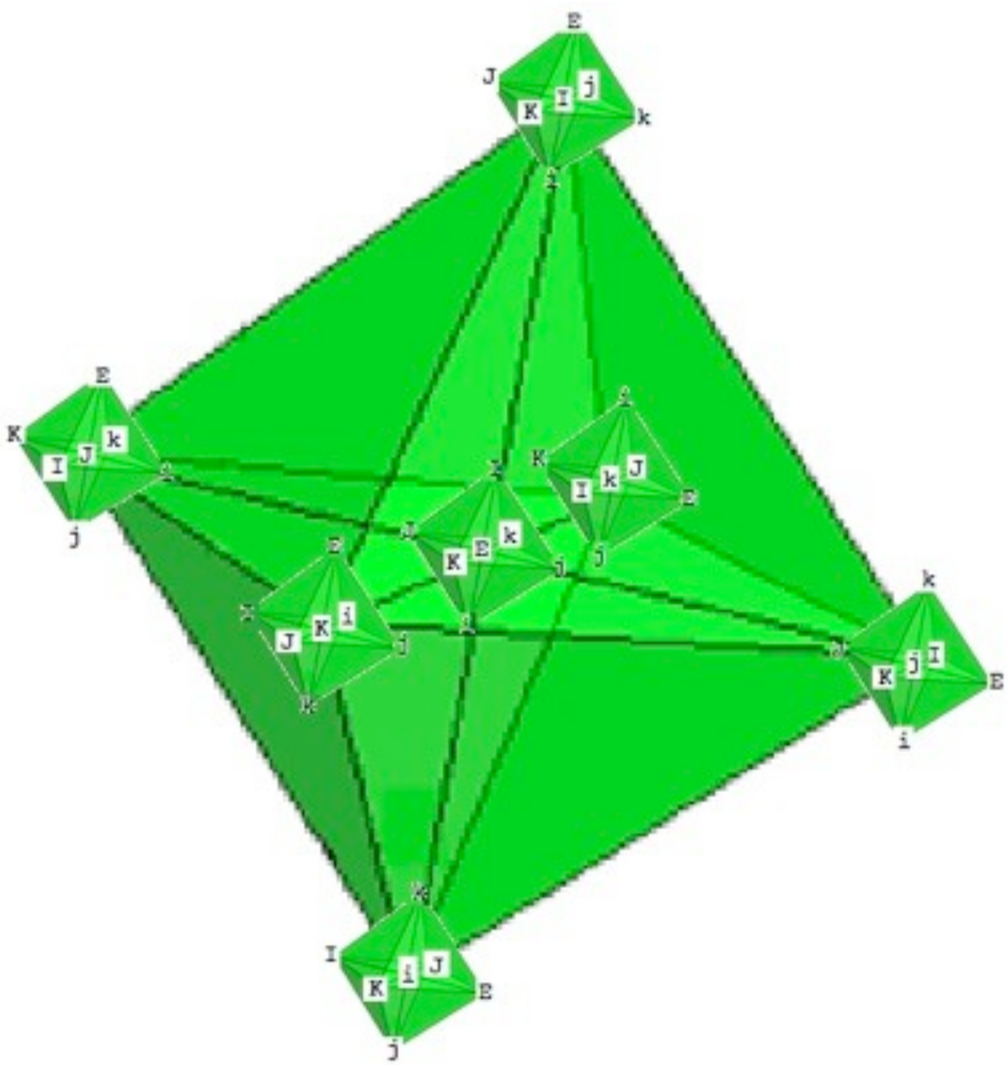


as, for example,

with one associative triple {i,j,k} being the vertices of a face of the Octahedron and the central element being {E}



in which alternate faces of the Octahedron correspond to associative triples. Since the Octonions have 7 associative triples, corresponding to the 7 Imaginary Octonions, the Heptaverton construction can be nested recursively

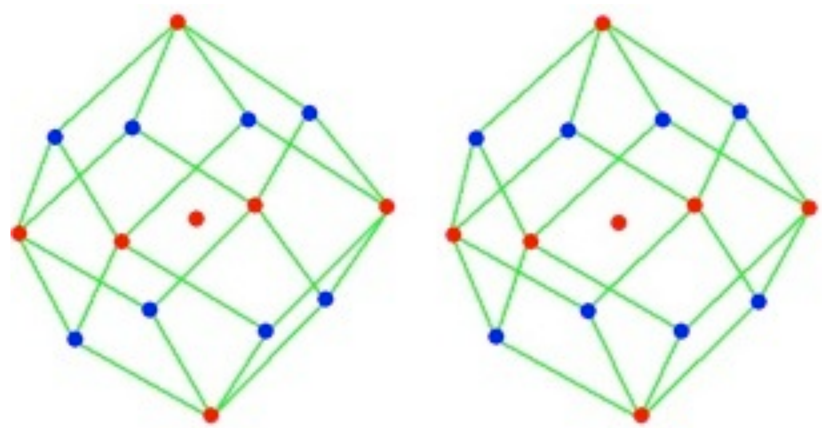


Note that the Octahedron is the Conformal Kepler representative of the two innermost planets Mercury and Venus.

The Octonion Recursive Structure corresponds to the 7 Imaginary Octonions, the 7 vertices of the Heptaverton, and the 7 Octonion Associative Triples. It is related to the Non-Unitary Physics of Octonions that is manifested in the Early Inflationary Phase of Our Universe.

Sedenions have 35 - 20 = 15 Excess Associative Triples,

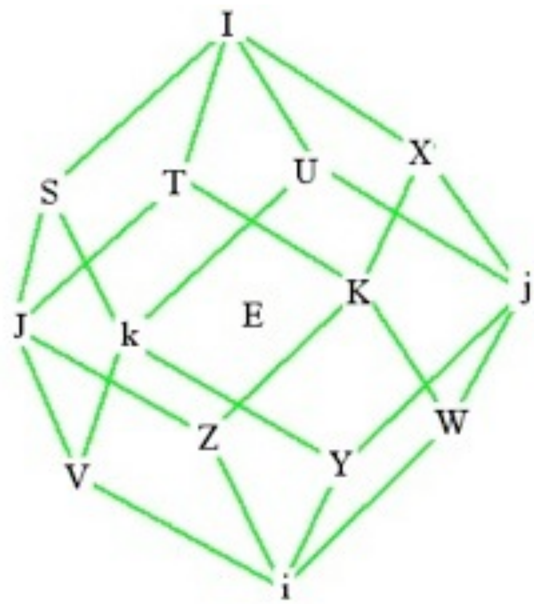
which correspond to the 15 Sedenion Imaginary basis elements { i, j, k, E, I, J, K, S, T, U, V, W, X, Y, Z } and to the Rhombic Dodecahedron



which has 14 vertices plus a center (shown above as a stereo-pair image) and so the natural polytope to use, with 7 of the vertices corresponding to 6 octahedral vertices plus the center correspond to

the 7 pure Imaginary Octonionic basis elements (red dots - octahedron vertices plus center) { i, j, k, E, I, J, K } that represent 7 Excess Associative Triples that are beyond the 28 of the two Irreducible Component copies of 14-dim G2 (including the 10-dim LieBall(7) at the core of G2 and the enhancing S3xS1 that extends LieBall(7) to G2) - in other words, they correspond to the 7 A_3 Octonion Associative Triples while the 28 = 2x14 = 2 x dim(G2) correspond to the 28 Associative Triples that newly emerge with at the A_4 level of Sedenions -

while the remaining 8 cube-type vertices correspond to the remaining 8 Sedenion basis elements (blue dots - cube vertices) { S, T, U, V, W, X, Y, Z } that represent two Irreducible Component copies of the non-core enhancing 4-dimensional S3 x S1 = SU(2) x U(1) = U(2) that extend/enhance the 10-dim LieBall(7) to 14-dim G2 .



The 35 Sedenion Associative Triples correspond to the 7 Octonion Associative Triples (and therefore to the 7 Imaginary Octonions)

plus the 28 generators of Spin(8) which correspond to
the ordinary 7-sphere S7 (and therefore to a second copy of the 7 Imaginary Octonions) plus
the 21 generators of Spin(7) which correspond to
the 14 generators of G2 (and therefore 14 of the Sedenion Zero Divisors) plus
the Spin(7)/G2 7-sphere with torsion S7# (and therefore the 7-dimensional representation of G2 and, equivalently, a second copy of half of the 14 G2 Sedenion Zero Divisors).

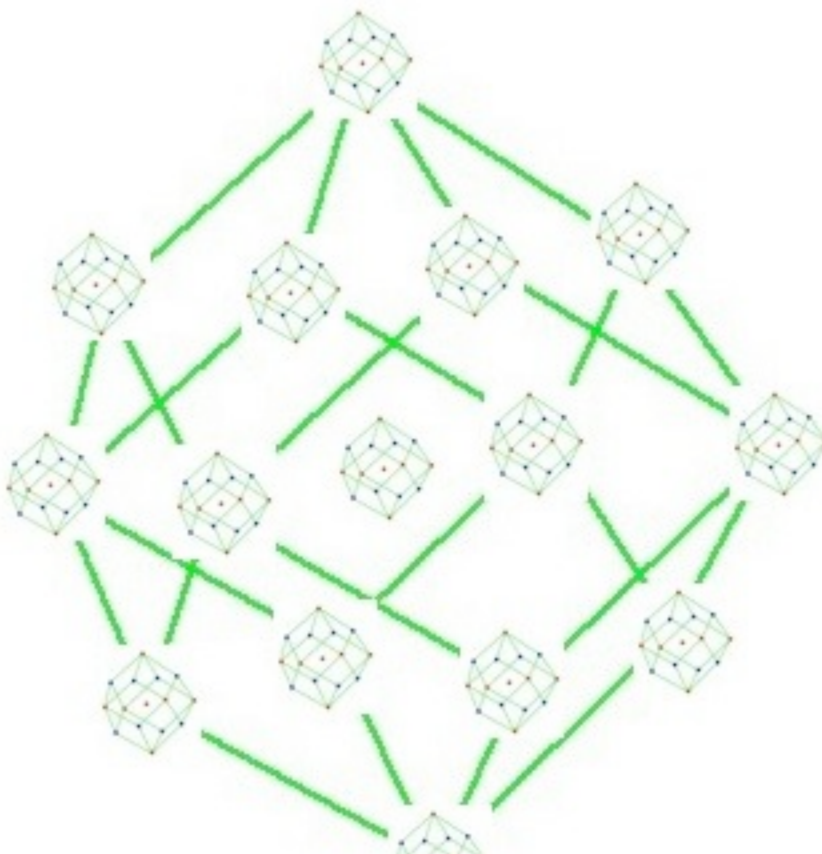
Let the Mirror Sedenion S that maps { i, j, k, E, I, J, K } <-> { T, U, V, W, X, Y, Z } represent the first 7 of the 14 G2 Zero Divisors and be at a Rhombic Dodecahedron Cube-Vertex opposite W

and the Mirror Sedenion E that maps { i, j, k } <-> { I, J, K } represent the 7 Spin(7)/G2 Zero Divisors and be at the center of the Rhombic Dodecahedron, and the Mirror Sedenion W that maps { T, U, V } <-> { X, Y, Z } represent second 7 of the 14 G2 Zero Divisors and be at a Rhombic Dodecahedron Cube-Vertex opposite S,

so that { S, E, W } represent the 21 Zero Divisor Sedenions of type Spin(7).

That leaves 3x4 = 12 Imaginary Sedenions { i, j, k } , { I, J, K } , { T, U, V } , and { X, Y, Z } which correspond to the remaining 12 vertices of the 14 vertices of the Rhombic Dodecahedron plus one central vertex,

so that, with { S, E, W } representing Zero Divisors, we can make a Sedenion Rhombic Dodecahedral Recursive Structure:



Note that the Rhombic Dodecahedron is the Conformal Kepler representative of the two outermost planets Uranus and Neptune.

The Sedenion Recursive Structure corresponds to the 15 Imaginary Sedenion and the 15 Vertices of the Rhombic Dodecahedron plus Center, but it does NOT involve all of the 35 Sedenion Associative Triples.

The 35 - 15 = 20 Sedenion Associative Triples that are not involved in the Sedenion Rhombic Dodecahedron plus Center Recursive Structure correspond to the 20 dimensions of the Sedenion Zero Divisor subspace represented by two copies of the 10-dimensional LieBall(7).

Each 10-dimensional LieBall(7) has the structure of the Conformal Space of dimension 7 = 5+2 over a 5-dimensional manifold that has the structure RP1 x S4 of the Shilov Boundary of the Bounded Complex Domain related to the compact Hermitian Symmetric Space Spin(7) / Spin(5) x Spin(2)) that has 21 - 10 - 1 = 10 Real dimensions and 10/2 = 5 Complex dimensions.

- The 10 dimensions of LieBall(7) correspond to the vector space of Spin(10).
- Spin(10) is the Conformal Group over 8-dimensional Spacetime, whose symmetry group Spin(8) is a subgroup of Spin(10).
- The two copies of the LieBall(7) in the Sedenions correspond to the two 28-dimensional components of the bosonic part of E8, each of which corresponds to a Spin(8) Lie algebra D4.
- The 5 dimensions of the related Shilov Boundary correspond to the rank of the Spin(10) Lie Algebra D5 as well as to the SU(5) subgroup of Spin(10).

288 and 22 Letters

There are 22 letters (not including Finals) in the Hebrew alphabet.

The 4-dimensional D4+ HyperDiamond Lattice of the 4-dimensional Physical Spacetime part of 8-dimensional Kaluza-Klein Spacetime has 288 vertices of in the layer of norm 22 distance from the origin (where the norm is the square norm usually used in lattice theory, that is, the inner product x.x of the vector x).

How to visualize the 288 vertices in layer 22.:

- Start with the 24 vertices of the 24-cell.
- Then consider 96 more vertices placed on each of the 96 edges of the 24-cell.
- Then consider 24 more vertices placed in each of the the 24 cells (octahedra) of the 24-cell.
- These 24 + 96 + 24 = 144 vertices correspond to the 144 vertices in each of layers 10, 17, and 20, and they correspond to half of the 288 vertices in layer 22.

Layer 22 with 288 vertices
follows
layer 21 with (1 + 3 + 7 + 21) x 8 = 256 = 16x16 = 2^8 vertices.

Vertices in some of the layers of the D4+ lattice (even-numbered in even D4 sublattice):

m=norm of layer	N(m) = no. vert.		
0	1	=	1
1	8	=	1 x 8
2	24	=	1 x 24
3	32	=	(1 + 3) x 8
4	24	=	1 x 24
5	48	=	(1 + 5) x 8
6	96	=	(1 + 3) x 24
7	64	=	(1 + 7) x 8
8	24	=	1 x 24
9	104	=	(1 + 3 + 9) x 8
10	144	=	(1 + 5) x 24
11	96	=	(1 + 11) x 8
12	96	=	(1 + 3) x 24
13	112	=	(1 + 13) x 8
14	192	=	(1 + 7) x 24
15	192	=	(1 + 3 + 5 + 15) x 8
16	24	=	1 x 24
17	144	=	(1 + 17) x 8
18	312	=	(1 + 3 + 9) x 24
19	160	=	(1 + 19) x 8
20	144	=	(1 + 5) x 24
21	256	=	(1 + 3 + 7 + 21) x 8
22	288	=	(1 + 11) x 24
23	192	=	(1 + 23) x 8
24	96	=	(1 + 3) x 24
25	248	=	(1 + 5 + 25) x 8
26	336	=	(1 + 13) x 24
27	320	=	(1 + 3 + 9 + 27) x 8
28	192	=	(1 + 7) x 24
29	240	=	(1 + 29) x 8
30	576	=	(1 + 3 + 5 + 15) x 24
31	256	=	(1 + 31) x 8
32	24	=	1 x 24
33	384	=	(1 + 3 + 11 + 33) x 8
34	432	=	(1 + 17) x 24
35	384	=	(1 + 5 + 7 + 35) x 8
36	312	=	(1 + 3 + 9) x 24
37	304	=	(1 + 37) x 8
38	480	=	(1 + 19) x 24
39	448	=	(1 + 3 + 13 + 39) x 8
40	144	=	(1 + 5) x 24
41	336	=	(1 + 41) x 8
42	768	=	(1 + 3 + 7 + 21) x 24
43	352	=	(1 + 43) x 8
44	288	=	(1 + 11) x 24
45	624	=	(1 + 3 + 5 + 9 + 15 + 45) x 8

The notation in the following table is based on the minimal norm of the D4 lattice being 1, in which case the D4 lattice is the lattice of integral quaternions. This is the second definition (equation 90) of the D4 lattice in Chapter 4 of Sphere Packings, Lattices, and Groups, 3rd edition, by Conway and Sloane (Springer 1999), who note that the Dn lattice is the checkerboard lattice in n dimensions.

m=norm of layer	N(m)=no. vert.	K(m)=N(m)/24
1	24	1
2	24	1
3	96	4
4	24	1
5	144	6
6	96	4
7	192	8
8	24	1
9	312	13
10	144	6
11	288	12
12	96	4
13	336	14
14	192	8
15	576	24
16	24	1
17	432	18
18	312	13
19	480	20
20	144	6
127	3,072	128
128	24	1
65,536=2^16	24	1
65,537	1,572,912	65,538
2,147,483,647	51,539,607,552	2,147,483,648
2,147,483,648=2^31	24	1

Perfect Numbers

The Perfect Numbers are numbers that are themselves the sum of their proper factors:

```
0 = 0
  = dimension of the D0 Lie Algebra of Spin(0) = {-1,+1}

1 = 1
  = dimension of the D1 Lie Algebra of Spin(2) = U(1)

6 = 1 + 2 + 3
  = dimension of the D2 Lie Algebra of
  Spin(4) = Spin(3)xSpin(3) = SU(2)xSU(2) = Sp(1)xSp(1) +
S3 x S3
(related to the Mersenne Prime 3 = 2^2 -1)

28 = 1 + 2 + 4 + 7 + 14
   = dimension of the D4 Lie Algebra of Spin(8)
(related to the Mersenne Prime 7 = 2^3 -1)

496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248
    = dimension of the D16 Lie Algebra of Spin(32)
(related to the Mersenne Prime 31 = 2^5 -1)

8,128 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 +
1,016 + 2,032 + 4,064
      = dimension of the D64 Lie Algebra of Spin(128)
(related to the Mersenne Prime 127 = 2^7 -1)

33,550,336
(related to the Mersenne Prime 8,191 = 2^13 -1)

and larger numbers
```

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