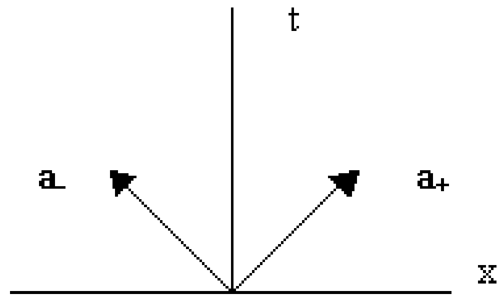
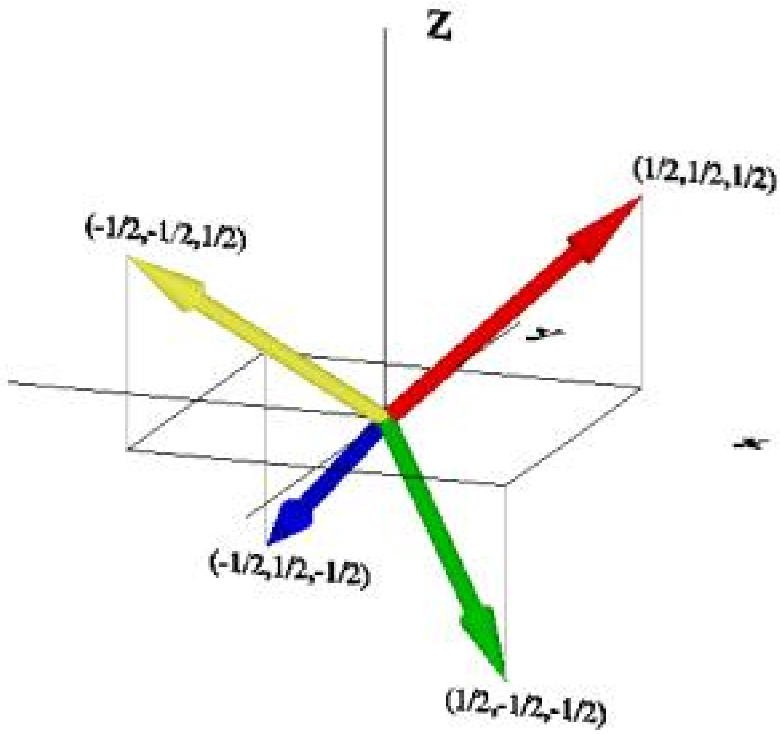
# The HyperDiamond Feynman Checkerboard in 1+3 dimensions reproduces the correct Dirac equation.

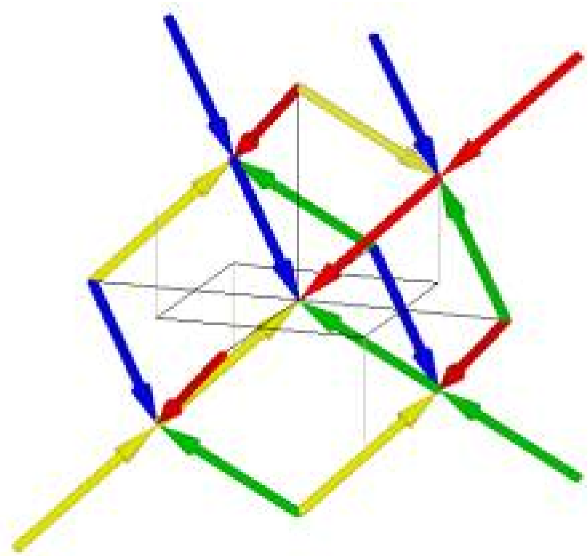
Urs Schreiber has done the work necessary for the proof, after reading the work of George Raetz presented on his web site. A very nice feature of the George Raetz web site is its illustrations, which include an image of a vertex of a 1+1 dimensional Feynman Checkerboard




and an image of a projection into three dimensions of a vertex of a 1+3 dimensional Feynman Checkerboard




and an image of flow contributions to a vertex in a HyperDiamond Random Walk from the four nearest neighbors in its past



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Urs Schreiber wrote on the subject: Re: Physically understanding the Dirac equation and 4D in the newsgroup sci.physics.research on 2002-04-03 19:44:31 PST (including an appended forwarded copy of an earlier post) and again on 2002-04-10 19:03:09 PST as found on the web page http://www-stud.uni-essen.de/~sb0264/spinors-Dirac-checkerboard.html and the following are excerpts from those posts:

> "... I know ... http://www.innerx.net/personal/tsmith/FynCkb.html ... and the corresponding lanl paper ...[ http://xxx.lanl.gov/abs/quant-ph/9503015 ]... and I know that Tony Smith does give a generalization of Feynman's summing prescription from 1+1 to 1+3 dimensions.
>
> But I have to say that I fail to see that this generalization reproduces the Dirac propagator in 1+3 dimensions, and that I did not find any proof that it does. Actually, I seem to have convinced myself that it does not, but I may of course be quite wrong. I therefore take this opportunity to state my understanding of these matters.
>
> First, I very briefly summarize (my understanding of) Tony Smith's construction:
>
> The starting point is the observation that the left |-> and right |+> going states of the 1+1 dim checkerboard model can be labeled by complex numbers
>
> |-> ---> (1 + i)
>
> |+> ---> (1 - i)
>
> (up to a factor) so that multiplication by the negative imaginary unit swaps components:

$(-i)\,(1 + i)/2 = (1 - i)/2$

$(-i)\,(1 - i)/2 = (1 + i)/2$ .

Since the path-sum of the 1+1 dim model reads

phi = sum over all possible paths of (-i eps m)^(number of bends of path) = sum over all possible paths of product over all steps of one path of -i eps m (if change of direction after this step generated by i) 1 (otherwise)

this makes it look very natural to identify the imaginary unit appearing in the sum over paths with the "generator" of kinks in the path. To generalize this to higher dimensions, more square roots of -1 are added, which gives the quaternion algebra in 1+3 dimensions. The two states |+> and |-> from above, which were identified with complex numbers, are now generalized to four states identified with the following quaternions (which can be identified with vectors in M^4 indicating the direction in which a given path is heading at one instant of time):

$(1 + i + j + k)$

$(1 + i - j - k)$

$(1 - i + j - k)$

$(1 - i - j + k)$ ,

which again constitute a (minimal) left ideal of the algebra (meaning that applying i,j, or k from the left on any linear combination of these four states gives another linear combination of these four states). Hence, now i,j,k are considered as "generators" of kinks in three spatial dimensions and the above summing prescription naturally generalizes to

phi = sum over all possible paths of   product over all steps of one path of

-i eps m (if change of direction after this step generated by i)

-j eps m (if change of direction after this step generated by j)

-k eps m (if change of direction after this step generated by k)

1 (otherwise)

The physical amplitude is taken to be

A * e^(i alpha)

where A is the norm of phi and alpha the angle it makes with the x0 axis.

As I said, this is merely my paraphrase of Tony Smith's proposal as I understand it.

I fully appreciate that the above construction is a nice (very "natural") generalization of the summing prescription of the 1+1 dim checkerboard model. But if it is to describe real fermions propagating in physical spacetime, this generalized path-sum has to reproduce the propagator obtained from the Dirac equation in 1+3 dimensions, which we know to correctly describe these fermions. Does it do that?

...

Hence I have taken a look at the material [that] ... George Raetz ... present[s] ... titled "The HyperDiamond Random Walk", found at http://www.pcisys.net/~bestwork.1/QRW/the_flow_quaternions.htm , which is mostly new to me.... I am posting this in order to make a suggestion for a more radical modification ...

[The]... equation ... DQ = (iE)Q ... is not covariant. That is because of that quaternion E sitting on the left of the spinor Q in the rhs of [the] equation ... . The Dirac operator D is covariant, but the unit quaternion E on the rhs refers to a specific frame. Under a Lorentz transformation L one finds

L DQ = iE LQ = L E' Q <=> DQ = E'Q

now with E' = L~ E L instead of E. This problem disappears when the unit quaternion E is brought to the *right* of the spinor Q. What we would want is an equation of the form

DQ = Q(iE) .

In fact, demanding that the spinor Q be an element of the minimal left ideal generated by the primitive projector

P = (1+y0)(1+E)/4 ,

so that

Q = Q' P ,

one sees that

DQ = Q(iE)

almost looks like the the *Dirac-Lanczos equation*. (See hep-ph/0112317, equation (5) or ... equation (9.36) [of]... W. Baylis, Clifford (Geometric) Algebras, Birkhaeuser (1996) ... ). To be equivalent to the Dirac-Lanczos equation, and hence to be correct, we need to require that

D = y0 @0 + y1 @1 + y2 @2 + y3 @3

instead of

... = @0 + e1 @1 + e2 @2 + e3 @3 .

All this amounts to sorting out in which particluar representation we are actually working here.

In an attempt to address these issues, I now redo the steps presented on http://www.pcisys.net/~bestwork.1/QRW/the_flow_quaternions.htm with some suitable modifications to arrive at the correct Dirac-Lanczos equation (this is supposed to be a suggestion subjected to discussion):

So consider a lattice in Minkoswki space generated by a unit cell spanned by the four (Clifford) vectors

r = (y0 + y1 + y2 + y3)/2

g = (y0 + y1 - y2 - y3)/2

b = (y0 - y1 + y2 - y3)/2

y = (y0 - y1 - y2 + y3)/2 .

(yi are the generators of the Dirac algebra {yi,yj} = diag(+1,-1,-1,-1)_ij.) This is Tony Smith's "hyper diamond". (Note that I use Clifford vectors instead of quaternions.) Now consider a "Clifford algebra-weighted" random walk along the edges of this lattice, which is described by four Clifford valued "amplitudes":

Kr, Kg, Kb, Ky

and such that

@r Kr = k (Kg y2 y3 + Kb y3 y1 + Ky y1 y2)

@b Kb = k (Ky y2 y3 + Kr y3 y1 + Kg y1 y2)

@g Kg = k (Kr y2 y3 + Ky y3 y1 + Kb y1 y2)

@y Ky = k (Kb y2 y3 + Kg y3 y1 + Kr y1 y2) .

(This is geometrically motivated. The generators on the rhs are those that rotate the unit vectors corresponding to the amplitudes into each other. "k" is some constant.) Note that I multiply the amplitudes from the *right* by the generators of rotation, instead of multiplying them from the left.

Next, assume that this coupled system of differential equations is solved by a spinor Q

Q = Q' (1+y0)(1+iE)/4

E = (y2 y3 + y3 y1 + y1 y2)/sqrt(3)

with

Kr = r Q

Kg = g Q

Kb = b Q

Ky = y Q .

This ansatz for solving the above system by means of a single spinor Q is, as I understand it, the central idea. But note that I have here modified it on the technical side: Q is explicitly an algebraic Clifford spinor in a definite minial left ideal, E squares to -1, not to +1, and the Ki are obtained from Q by premultiplying with the Clifford basis vectors defined above.

Substituting this ansatz into the above coupled system of differential equations one can form one covariant expression by summing up all four equations:

(r @r + g @g + b @b + y @y) Q = k sqrt(3) Q E

The left hand side is immediate. To see that the right hand side comes out as indicated simply note that r + g + b + y = y0 and that Q y0 = Q by construction.

The above equation is the Dirac-Lanczos-Hestenes-Guersey equation, the algebraic version of the equation describing the free relativistic electron. The left hand side is the flat Dirac operator r @r + g @g + b @b + y @y = ym @m and the right hand side, with k = mc / (hbar sqrt(3)) , is equal to the mass term i mc / hbar Q.

As usual, there are a multitude of ways to rewrite this. If one wants to emphasize biquaternions then premultiplying everything with y0 and splitting off the projector P on the right of Q to express everything in terms of the, then also biquaternionic, Q' (compare the definitions given above) gives Lanczos' version (also used by Baylis and others).

I think this presentation improves a little on that given on George Raetz's web site: The factor E on the right hand side of the equation is no longer a nuisance but a necessity. Everything is manifestly covariant (if one recalls that algebraic spinors are manifestly covariant when nothing non-covariand stands on their *left* side). The role of the quaternionic structure is clarified, the construction itself does not depend on it. Also, it is obvious how to generalize to arbitrary dimensions. In fact, one may easily check that for 1+1 dimensions the above scheme reproduces the Feynman model.

While I enjoy this, there is still some scepticism in order as long as a central questions remains to be clarified: How much of the Ansatz K(r,g,b,y) = (r,g,b,y) Q is whishful thinking?

For sure, **every Q that solves the system of coupled differential equations that describe the amplitude of the random walk on the hyper diamond lattice also solves the Dirac equation**.

But what about the other way round?

**Does every Q that solves the Dirac equation also describe such a random walk. ...".**

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My proposal to answer the question raised by Urs Schreiber

**Does every solution of the Dirac equation also describe a HyperDiamond Feynman Checkerboard random walk?**

uses symmetry.

The hyperdiamond random walk transformations include the transformations of the Conformal Group:

- rotations and boosts (to the accuracy of lattice spacing);
- translations (to the accuracy of lattice spacing);
- scale dilatations (to the accuracy of lattice spacing): and
- special conformal transformations (to the accuracy of lattice spacing).

Therefore, to the accuracy of lattice spacing, the hyperdiamond random walks give you all the conformal group Dirac solutions, and since the full symmetry group of the Dirac equation is the conformal group, the answer to the question is "Yes". Thanks to the work of Urs Schreiber:

## The HyperDiamond Feynman Checkerboard in 1+3 dimensions does reproduce the correct Dirac equation.

Here are some references to the conformal symmetry of the Dirac equation:

R. S. Krausshar and John Ryan in their paper Some Conformally Flat Spin Manifolds, Dirac Operators and Automorphic Forms at math.AP/0212086 say:

> "... In this paper we study Clifford and harmonic analysis on some conformal flat spin manifolds. ... manifolds treated here include RPn and S1 x S(n-1). Special kinds of Clifford-analytic automorphic forms associated to the different choices of are used to construct Cauchy kernels, Cauchy Integral formulas, Green's kernels and formulas together with Hardy spaces and Plemelj projection operators for Lp spaces of hypersurfaces lying in these manifolds. ... **Solutions to the Dirac equation** are called Clifford holomorphic functions or monogenic functions. Such functions **are covariant under ... conformal or .... Mobius transformations acting over Rn u {oo}**. ...".

Barut and Raczka, in their book Theory of Group Representations and Applications (World 1986), say, in section 21.3.E, at pages 616-617:

> "... E. The Dynamical Group Interpretation of Wave Equations.
>
> ... Example 1. **Let G = O(4,2)**. Take U to be the 4-dimensional non-unitary representation in which the generators of G are given in terms of the 16 elements of the algebra of Dirac

> matrices as in exercise 13.6.4.1. Because (1/2)L_56 = gamma_0 has eigenvalues n = +/-1, taking the simplest mass relation mn = K, we can write
>
> (m gamma_0 - K) PSI(dotp) = 0, where K is a fixed constant.
>
> **Transforming this equation** with the Lorentz transformation of parameter E
>
> PSI(p) = exp(i E N) PSI(p)
>
> N = (1/2) gamma_0 gamma
>
> **gives**
>
> (gamma^u p_u - K) PSI(p) = 0
>
> which is **the Dirac equation** ...".

P. A. M. Dirac, in his paper Wave Equations in Conformal Space, Ann. Math. 37 (1936) 429-442, reprinted in The Collected Works of P. A. M. Dirac: Volume 1: 1924-1948, by P. A. M. Dirac (author), Richard Henry Dalitz (editor), Cambridge University Press (1995), at pages 823-836, said:

> "... by passing to a four-dimensional conformal space ... a ... greater symmetry of ... equations of physics ... is shown up, and their invariance under a wider group is demonstrated. ... The spin wave equation ... seems to be the only simple conformally invariant wave equation involving the spin matrices. ... This equation is equivalent to the usual wave equation for the electron, except ...[that it is multiplied by]... the factor (1 + alpha_5) , which introduces a degeneracy. ...".

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Tony's Home

This paper is also at CERN-CDS-EXT-2004-030.