E8 Geometry and Physics

(E8, the Lie algebra of an E8 Physics Model, is rank 8 and has 8+240 = 248 dimensions - Compact Version - Euclidean Signature - for clarity of exposition - much of this is from the book Einstein Manifolds (by Arthur L. Besse, Springer-Verlag 1987):

Type EVIII rank 8 Symmetric Space Rosenfeld's Elliptic Projective Plane (OxO)P2

E8 / Spin(16) = 64 + 64

The Octonionic structure of (OxO)P2 gives it a natural torsion structure *

for which 64 looks like (8 fermion particles) x (8 Dirac Gammas)

and 64 looks like (8 fermion antiparticles) x (8 Dirac Gammas)

Type BDI(8,8) rank 8 Symmetric Space real 8-Grassmannian manifold of R16 or set of the RP7 in RP15

Spin(16) / (Spin(8) x Spin(8)) = 64

Spin(16) is rank 8 and has 8+112 = 120 dimensions and looks like a 64-dim Base Manifold

whose curvature is determined by a 28+28=56-dim Gauge Group Spin(8) x Spin(8)

The 64-dim Base Manifold looks like (8-dim Kaluza-Klein spacetime) x (8 Dirac Gammas)

Due to the special isomorphisms Spin(6) = SU(4) and Spin(2) = U(1) and the topological equality RP1 = S1

Spin(8) / (Spin(6) x Spin(2)) = real 2-Grassmannian manifold of R8 or set of the RP1 in RP7

Spin(6) gives Conformal MacDowell-Mansouri Gravity

Spin(8) / U(4) = Spin(8) / SU(4) x U(1) = set of metric-compatible fibrations S1 -> RP7 -> CP3

SU(4) / SU(3)xU(1) = CP3

SU(3) gives color force

U(1) gives electromagnetism

CP3 contains CP2 = SU(3) / U(1) x SU(2) and so gives SU(2) weak force

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Torsion and E8 / Spin(16) = 64+64

Martin Cederwall and Jakob Palmkvist, in "The octic E8 invariant" hep-th/0702024, say:

> "... The largest of the finite-dimensional exceptional Lie groups, E8, with Lie algebra e8, is an interesting object ... its root lattice is the unique even self-dual lattice in eight dimensions (in euclidean space, even self-dual lattices only exist in dimension 8n). ... Because of self-duality, there is only one conjugacy class of representations, the weight lattice equals the root lattice, and there is no "fundamental" representation smaller than the adjoint. ... Anything resembling a tensor formalism is completely lacking. A basic ingredient in a tensor calculus is a set of invariant tensors, or "Clebsch&endash;Gordan coefficients". The only invariant tensors that are known explicitly for E8 are the Killing metric and the structure constants ...
>
> The goal of this paper is to take a first step towards a tensor formalism for E8 by explicitly constructing an invariant tensor with eight symmetric adjoint indices. ... On the mathematical side, the disturbing absence of a concrete expression for this tensor is unique among the finite-dimensional Lie groups. Even for the smaller exceptional algebras g2, f4, e6 and e7, all invariant tensors are accessible in explicit forms, due to the existence of "fundamental" representations smaller than the adjoint and to the connections with octonions and Jordan algebras. ...
>
> The orders of Casimir invariants are known for all finite-dimensional semi-simple Lie algebras. They are polynomials in U(g), the universal enveloping algebra of g, of the form t_(A1...Ak) T^(A1 . . . TAk), where t is a symmetric invariant tensor and T are generators of the algebra, and they generate the center U(g)^(g) of U(g). The Harish-Chandra homomorphism is the restriction of an element in U(g)^(g) to a polynomial in the Cartan subalgebra h, which will be invariant under the Weyl group W(g) of g. Due to the fact that the Harish-Chandra homomorphism is an isomorphism from U(g)^(g) to U(h) W(g) one may equivalently consider finding a basis of generators for the latter, a much easier problem. The orders of the invariants follow more or less directly from a diagonalisation of the Coxeter element, the product of the simple Weyl reflections ...
>
> In the case of e8, the center U(e8)^(e8) of the universal enveloping subalgebra is generated by elements of orders 2, 8, 12, 14, 18, 20, 24 and 30. The quadratic and octic invariants correspond to primitive invariant tensors in terms of which the higher ones should be expressible. ... the explicit form of the octic invariant is previously not known ...
>
> E8 has a number of maximal subgroups, but one of them, Spin(16)/Z2, is natural for several reasons. Considering calculational complexity, this is the subgroup that leads to the smallest number of terms in the Ansatz. Considering the connection to the Harish-Chandra homomorphism, K = Spin(16)/Z2 is the maximal compact subgroup of the split form G = E8(8). The Weyl group is a discrete subgroup of K, and the Cartan subalgebra h lies entirely in the coset directions g/k ...
>
> We thus consider the decomposition of the adjoint representation of E8 into representations of the maximal subgroup Spin(16)/Z2. The adjoint decomposes into the adjoint 120 and a chiral spinor 128. ...
>
> Our convention for chirality is GAMMA_(a1...a16) PHI = + e_(a1...a16) PHI . The e8 algebra becomes
> (2.1)

[T^(ab) , T^(cd)] = 2 delta^([a)_([c) T^(b])_(d]) ,

[T^(ab) , PHI^(alpha)] = (1/4) (GAMMA^(ab) PHI)^(alpha) ,

[PHI^(alpha) , PHI^(alpha)] = (1/8) (GAMMA_(ab))^(alpha beta) T^(ab) ,

... The coefficients in the first and second commutators are related by the so(16) algebra. The normalisation of the last commutator is free, but is fixed by the choice for the quadratic invariant, which for the case above is

X2 = (1/2) T_(ab) T^(ab) + PHI_(alpha) PHI^(alpha) .

Spinor and vector indices are raised and lowered with delta . Equation (2.1) describes the compact real form, E8(-248) .

By letting PHI -> i PHI one gets E8(8), where the spinor generators are non-compact, which is the real form relevant as duality symmetry in three dimensions (other real forms contain a non-compact Spin(16)/Z2 subgroup).

The Jacobi identities are satisfied thanks to the Fierz identity

(GAMMA_(ab)_[(alpha beta) (GAMMA_(ab)_(alpha beta)] = 0 ,

which is satisfied for so(8) with chiral spinors, so(9), and so(16) with chiral spinors

(in the former cases the algebras are so(9), due to triality, and f4).

The Harish-Chandra homomorphism tells us that the "heart" of the invariant lies in an octic Weyl-invariant of the Cartan subalgebra. A first step may be to lift it to a unique Spin(16)/Z2-invariant in the spinor, corresponding to applying the isomorphism fÅ|1 above. It is gratifying to verify ... that there is indeed an octic invariant (other than (PHI PHI)^4), and that no such invariant exists at lower order. ...

Forming an element of an irreducible representation containing a number of spinors involves symmetrisations and subtraction of traces, which can be rather complicated. This becomes even more pronounced when we are dealing with transformation ... under the spinor generators, which will transform as spinors. Then irreducibility also involves gamma-trace conditions. ...

The transformation ... under the action of the spinorial generator is an so(16) spinor. The vanishing of this spinor is equivalent to e8 invariance. The spinorial generator acts similarly to a supersymmetry generator on a superfield ...

The final result for the octic invariant is, up to an overall multiplicative constant:

$$
\begin{aligned}
X_8 = {}& \tfrac{1}{3072}\varepsilon^{a_1\ldots a_{16}}T_{a_1a_2}\ldots T_{a_{15}a_{16}} \\
&\quad - 30\mathrm{tr}T^8 + 14\mathrm{tr}T^6\mathrm{tr}T^2 + \tfrac{35}{4}(\mathrm{tr}T^4)^2 - \tfrac{35}{8}\mathrm{tr}T^4(\mathrm{tr}T^2)^2 + \tfrac{15}{64}(\mathrm{tr}T^2)^4 \\
& + \big[2\mathrm{tr}T^6 - \mathrm{tr}T^4\mathrm{tr}T^2 + \tfrac{1}{8}(\mathrm{tr}T^2)^3\big](\phi\phi) \\
&\quad + \big[\big(\tfrac{5}{4}\mathrm{tr}T^4 - \tfrac{1}{2}(\mathrm{tr}T^2)^2\big)T^{ab}T^{cd} + \tfrac{27}{4}\mathrm{tr}T^2T^{ab}(T^3)^{cd} \\
&\qquad - 15T^{ab}(T^5)^{cd} - 15(T^3)^{ab}(T^3)^{cd}\big](\phi\Gamma_{abcd}\phi) \\
&\quad + \big[\tfrac{1}{16}\mathrm{tr}T^2T^{ab}T^{cd}T^{ef}T^{gh} - \tfrac{5}{8}T^{ab}T^{cd}T^{ef}(T^3)^{gh}\big](\phi\Gamma_{abcdefgh}\phi) \\
&\quad - \tfrac{1}{192}T^{ab}T^{cd}T^{ef}T^{gh}T^{ij}T^{kl}(\phi\Gamma_{abcdefghijkl}\phi) \\
& + \big[7\mathrm{tr}T^4 - \tfrac{31}{8}(\mathrm{tr}T^2)^2\big](\phi\phi)^2 \\
&\quad - \tfrac{3}{64}T^{ab}T^{cd}T^{ef}T^{gh}(\phi\phi)(\phi\Gamma_{abcdefgh}\phi) \\
&\quad + \big[\tfrac{5}{64}T^{ab}T^{cd}T^{ef}T^{gh} - \tfrac{15}{16}T^{ab}T^{ce}T^{df}T^{gh} \\
&\qquad + \tfrac{5}{8}T^{ae}T^{bf}T^{cg}T^{dh}\big](\phi\Gamma_{abcd}\phi)(\phi\Gamma_{efgh}\phi) \\
&\quad + \big[\tfrac{3}{2}(T^3)^{ab}T^{cd} - \tfrac{1}{8}\mathrm{tr}T^2T^{ab}T^{cd}\big](\phi\phi)(\phi\Gamma_{abcd}\phi) \\
&\quad + \big[\tfrac{15}{16}(T^3)^{ab}T^{cd} - \tfrac{3}{16}\mathrm{tr}T^2T^{ab}T^{cd} + \tfrac{5}{4}(T^2)^{ac}(T^2)^{bd}\big](\phi\Gamma_{ab}{}^{ij}\phi)(\phi\Gamma_{cdij}\phi) \\
&\quad + \tfrac{15}{8}T^{ab}T^{cd}(T^2)^{ef}(\phi\Gamma_{abe}{}^{i}\phi)(\phi\Gamma_{cdfi}\phi) \\
& + \tfrac{1}{2}\mathrm{tr}T^2(\phi\phi)^3 + \tfrac{55}{32}T^{ab}T^{cd}(\phi\phi)^2(\phi\Gamma_{abcd}\phi) \\
&\quad + \tfrac{1}{8}T^{ab}T^{cd}(\phi\phi)(\phi\Gamma_{ab}{}^{ij}\phi)(\phi\Gamma_{cdij}\phi) \\
&\quad + \big[-\tfrac{1}{384}T^{ab}T^{cd} + \tfrac{7}{192}T^{ac}T^{bd}\big](\phi\Gamma_{ab}{}^{ij}\phi)(\phi\Gamma_{cd}{}^{kl}\phi)(\phi\Gamma_{ijkl}\phi) \\
& - \tfrac{57}{32}(\phi\phi)^4 + \tfrac{1}{12288}(\phi\Gamma_{ab}{}^{cd}\phi)(\phi\Gamma_{cd}{}^{ef}\phi)(\phi\Gamma_{ef}{}^{gh}\phi)(\phi\Gamma_{gh}{}^{ab}\phi) \\
& + \beta[-\tfrac{1}{2}\mathrm{tr}T^2 + (\phi\phi)]^4 \ .
\end{aligned}
\tag{2.3}
$$

Here, β is an arbitrary constant multiplying the fourth power of the quadratic invariant. The trace vanishes for $\beta = \frac{9}{127}$ (that such a value exists at all is non-trivial and provides a further check on the coefficients). The occurrence of the prime 127 is not incidental; taking the trace of $\delta^{(AB}\delta^{CD}\delta^{EF}\delta^{GH)}$ gives $\delta_{GH}\delta^{(AB}\delta^{CD}\delta^{EF}\delta^{GH)} = (\frac{1}{7}\cdot 248 + \frac{6}{7})\delta^{(AB}\delta^{CD}\delta^{EF)} = \frac{2\cdot 127}{7}\delta^{(AB}\delta^{CD}\delta^{EF)}$. The actual technique we use for calculating the trace is not to extract the eight-index tensor, but to act on the invariant with $\frac{1}{2}\frac{\partial}{\partial T_{ab}}\frac{\partial}{\partial T^{ab}} + \frac{\partial}{\partial \phi_\alpha}\frac{\partial}{\partial \phi^\alpha}$. We remind that eq. (2.3) gives the octic invariant for the compact form $E_{8(-248)}$. The corresponding expression for the split form $E_{8(8)}$ is obtained by a sign change of the terms containing ϕ^{4k+2}.

...".

Martin Cederwall, in hep-th/9310115, says:

"... The only simply connected compact parallelizable manifolds are the Lie groups and S7. If these vectorfields exist one can use them to define parallel transport of vectors. Since transport around any closed curve gives back the same vector, the curvature of the corresponding connection vanishes. We can think of the manifold equipped with this connection as "flat", and the transport as translation.

If the parallelizing connection is written as GAMMA~ = GAMMA - T where GAMMA is the metric connection, the vielbeins will not be covariantly constant, but transport as De = T (T is torsion, and this can be taken as its definition). Then ...

[D_a , D_b] = 2 T_ab^c D_c

... These are our S7 transformations ... What distinguishes S7 from the Lie groups is that its torsion ("structure constants") vary over the space. ... ".

Martin Cederwall and Christian R. Preitschopf, in hep-th/0702024, say:

"... it is the non-associativity of O that is responsible for the non-constancy of the torsion tensor [for S7] (while the non-commutativity accounts for its non-vanishing) and for the necessity of utilizing inequivalent products associated with different points X Å¸S7. We call this field-dependent multiplication the X-product.

One should note that the transformation ...[for S7]... relies on the transformation of the parameter field X ... while for group manifolds (and thus for the lower-dimensional spheres S1 and S3 associated with C and H) ... [the transformation is independent of a parameter field]... transform independently. A consequence is that fermions cannot transform without the presence of a parameter field, since a fermionic octonion is not invertible. ... Fermions, due to non-invertibility, can be assigned to endpoints of the diagram only; no path may pass via a fermion. ...

We call a field (bosonic or fermionic) transforming according to ...[the X-product]... a spinor under S7. ...

Let r, s, ... be S7 spinors ... Can this representation be formed as a tensor product of spinor representations? Due to the non-linearity, the answer is no.... we can form spinors as trilinears of spinors u = (r ox s*) ox t , and in this way only. ...

It should be possible to realize E6 = SL(3;O) ... on them in a "spinor-like" manner, much like SO(10) = SL(2; O) acts on its 16-dimensional spinor representations that play the role of homogeneous coordinates for OP1 ...

That would open for for a twistor transform ... for elements in J3(O) (the exceptional Jordan algebra of 3x3 hermitiean octonionic matrices) with zero Freudenthal product - a known realization of OP2. Then one would have a direct analogy to the twistor transform of the masslessness condition in SL(2;O) that leads to OP1 as the projective light-cone ...

we would like to address the question of anomaly cancellation: under what circumstances is the Schwinger term "quantum mechanically consistent", i.e. when is the BRST operator quantum mechanically nilpotent, and what actual exact form of the Schwinger term is needed? ... to construct a (classical) BRST operator for the S7 algebra with field-dependent structure functions ... turns out to be extremely simple. The BRST operator takes the same form as for a Lie algebra, namely

$$Q = c^i J_i - T_{ij}^k(X) c^i c^j b_k$$

where b_i and c^i are fermionic ghosts ... Higher order ghost terms are not present since the Jacobi identities hold ... This makes BRST analysis quite manageable. ...

Then, turning to ... the quantum algebra, ... We have ... demonstrated the non-trivial fact that Q may be nilpotent, and that ... non-trivial central extensions ...[of S7]... or Schwinger terms ... may be used as a gauge algebra. Normally, one would have expected $Q^2=0$ to put a constraint on the number of transforming octonionic fields, but that is not the case at hand. Instead one is permitted, for any field content, to adjust the numerical coefficient ... in J in order to fulfil that relation ...

It seems that ... the S7 or ... non-trivial central extensions ...[of S7]... or Schwinger terms ...ghosts do not come in an S7 representation. This is also confirmed by an attempt to construct a representation (other than scalar) for imaginary octonions, which turns out to be impossible. ...

A part of the structure of S7 we have treated only fragmentarily is representation theory. ... It is not immediately clear even how to define a representation. We have quite strong feelings, though, that the spinorial representations and the adjoint, as described in this paper, in some sense are the only ones allowed, and that the spinor representation is the only one to which a variable freely can be assigned. ...".
