# E6 in E8, PSL(2,11) and E8(p)

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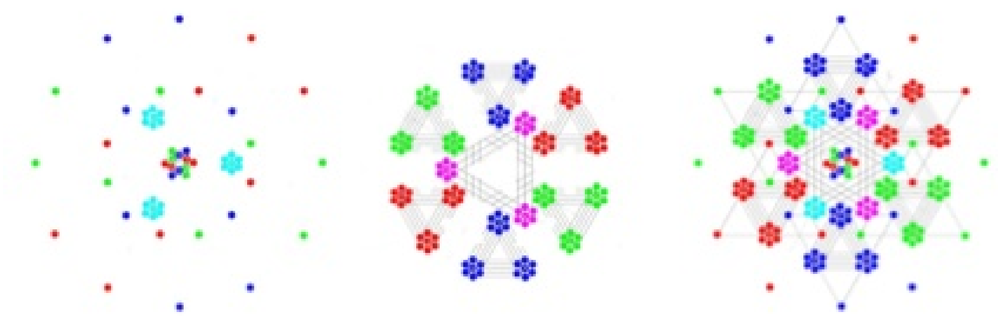
## E6 in E8

Note that in the below images some of the 240 E8(8) vertices are projected to the same point, so that when counting root vectors keep in mind:

- each of the vertices in the center with white dots are points to which 3 vertices are projected, so that each of the 6 circles with a white dot represents 3 vertices;
- each of the vertices surrounded by 6 same-color nearest neighbors with yellow dots are points to which 2 vertices are projected, so that each of the 24 circles with a yellow dot represents 2 vertices.

The right figure in the image below shows the 240 root vectors of 248-dimensional E8:



The left figure in the image above shows the 72 root vectors of 78-dimensional E6 which is made up of:

- 28-dimensional D4 (24 cyan root vectors)
- ( 8+8 ) complex D4 vectors ( 8+8 blue root vectors )
- 1 Cartan subalgebra element for complexification of D4 vectors
- ( 8+8 ) complex D4 +half-spinors( 8+8 red root vectors )
- ( 8+8 ) complex D4 -half-spinors ( 8+8 green root vectors )
- 1 Cartan subalgebra element for complexification of D4 spinors

Given a basis {1,i} of the complex numbers, the 3 sets of 8+8 in E6 can each be regarded as representing 8 complex elements of the form

$$8 \times 1 + 8 \times i$$

so that the representation spaces of 8-dimensional Kaluza-Klein spacetime and the 8 fundamental first-generation fermion particles and the 8 fundamental first-generation fermion antiparticles can be seen as complex as is useful for calculation of particle masses and force strength constants using an approach motivated by that of Armand Wyler.

To see how to expand from E6 to E8, consider that E8 has octonionic structure, evidenced by the fact that E8 / Spin(16) = (OxO)P2 = Rosenfeld's ocoto-octonionic projective plane., so that E6 must be "Octonified" as follows:

- First, consider the D4 part of E6, which is not explicitly complexified, so it must be extended to operate on the octonions of E8. Ignoring signature subtleties, E6 has one D4 = Spin(8), whose action must be extended to octonion space. Consider the full spinor representation of Spin(8). According to F. Reese Harvey in his book "Spinors and Calibrations" (Academic 1990 at page 287): "... Spin(8) acts transitively on S7 x S7 ...", where each of the two S7 are the unit sphere in each of the 8-dimensional half-spinor representation spaces of Spin(8).

  So, to expand to E8, each of the S7 must be Octonified. This is done by introducing an octonion product among the points of each S7. Unlike S3 with a quaternion product that closes to form a Lie group, S7 under an octonion product does not close, but expands to form a 28-dimensional Spin(8) that can be seen as an S7, another S7, and a 14-dimensional G2. Since each of the two S7 expands to a Spin(8):

  Expanding E6 to E8 goes from the one D4 in E6 to 2 D4 in E8. The 24 root vectors of the second D4 are the 24 magenta root vectors in the central figure of the above image.
- Second, consider each of the 3 ( blue, red, and green in the E6 left figure of the above image ) sets of 8+8 root vectors in E6 with complex form 8 x 1 + 8 x e ( for complex basis here denoted {1,e} )

  To Octonify them they must be expanded from complex with basis {1,e} to octonion with basis {1,i,j,k,e,ie,je, ke}

  by adding 6 more root vectors ( {i,j,k} added corresponding to {1} and {ie,je,ke} added corresponding to {e} ) such that the each of the 3 sets of 8+8 = 16 can, when expanded to E8, each be regarded a representing 8+8+8+8 +8+8+8+8 = 64 octonionic elements of the form

  8 x 1 + 8 x i + 8 x j + 8 x k + 8 x e + 8 x ie + 8 x je + 8 x ke

  by adding 6 new sets of 8 root vectors for each of the vector blue, +half-spinor red, and -half-spinor green as shown in the central figure of the above image, for a total of 3 x 6x8 = 3x48 = 6 x 24 =144 of the root vectors in the central figure of the above image.

( Note that, since the complex structure of E6 remains implicitly in the structure of E8, it is still available for use by Armand Wyler-type approaches ( such as I use in my model ) for calculation of force strengths, particle masses, etc. )

**So, to expand the 72 root vectors of 78-dimensional E6 to the 240 root vectors of 248-dimensional E8,**

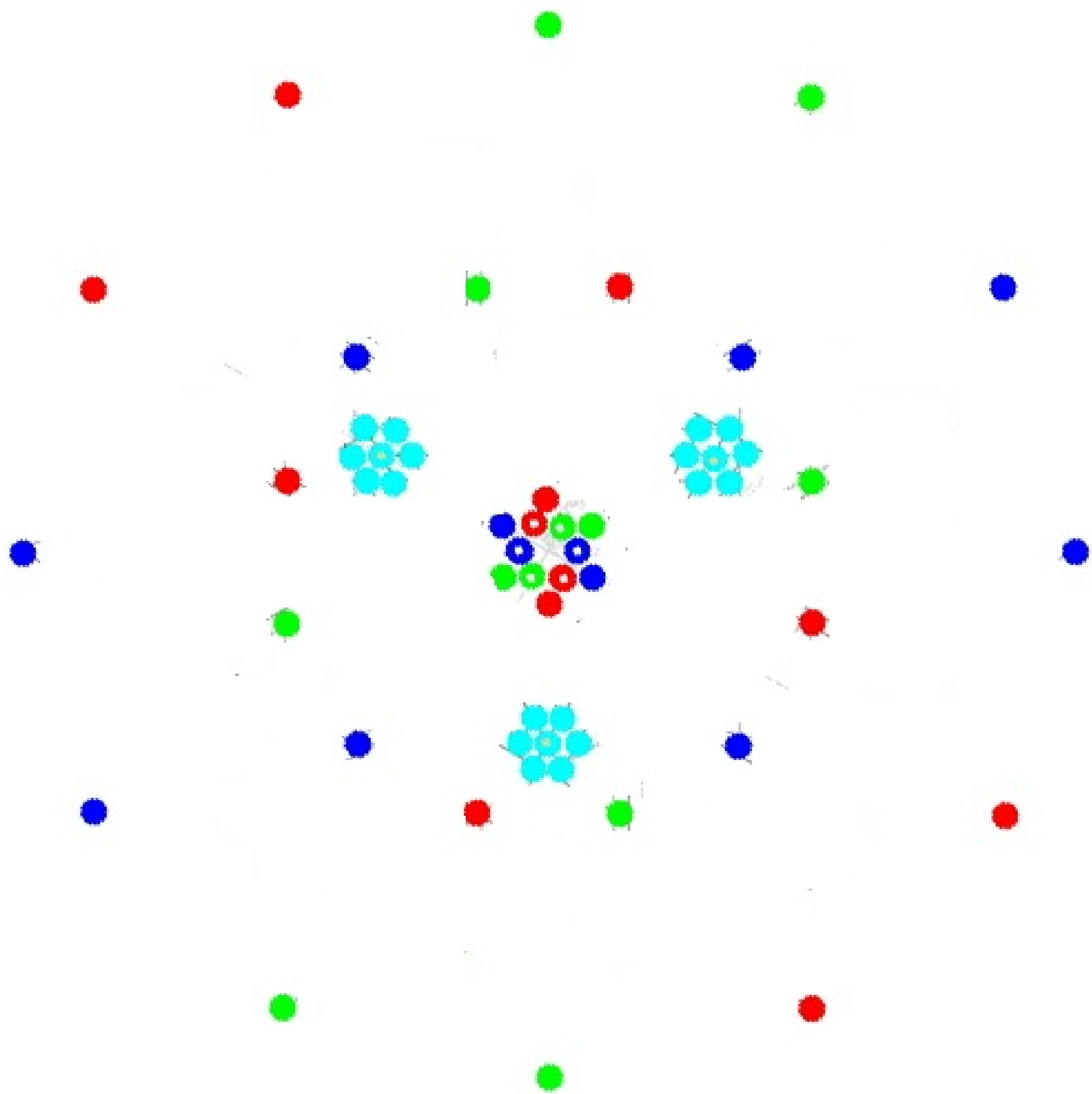
**add the 72 root vectors of the left figure of the above image**

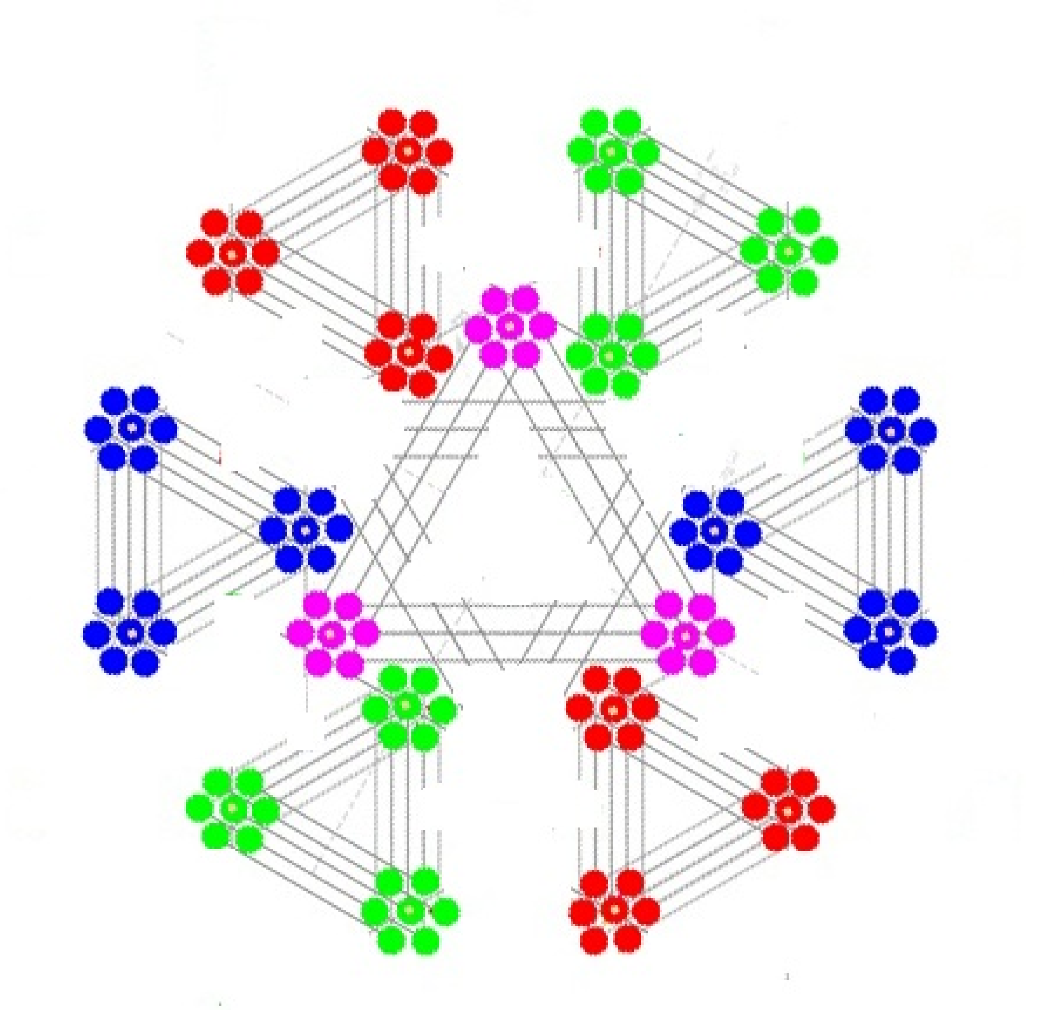
**to the 24+144 = 168 root vectors of the central figure of the above image**

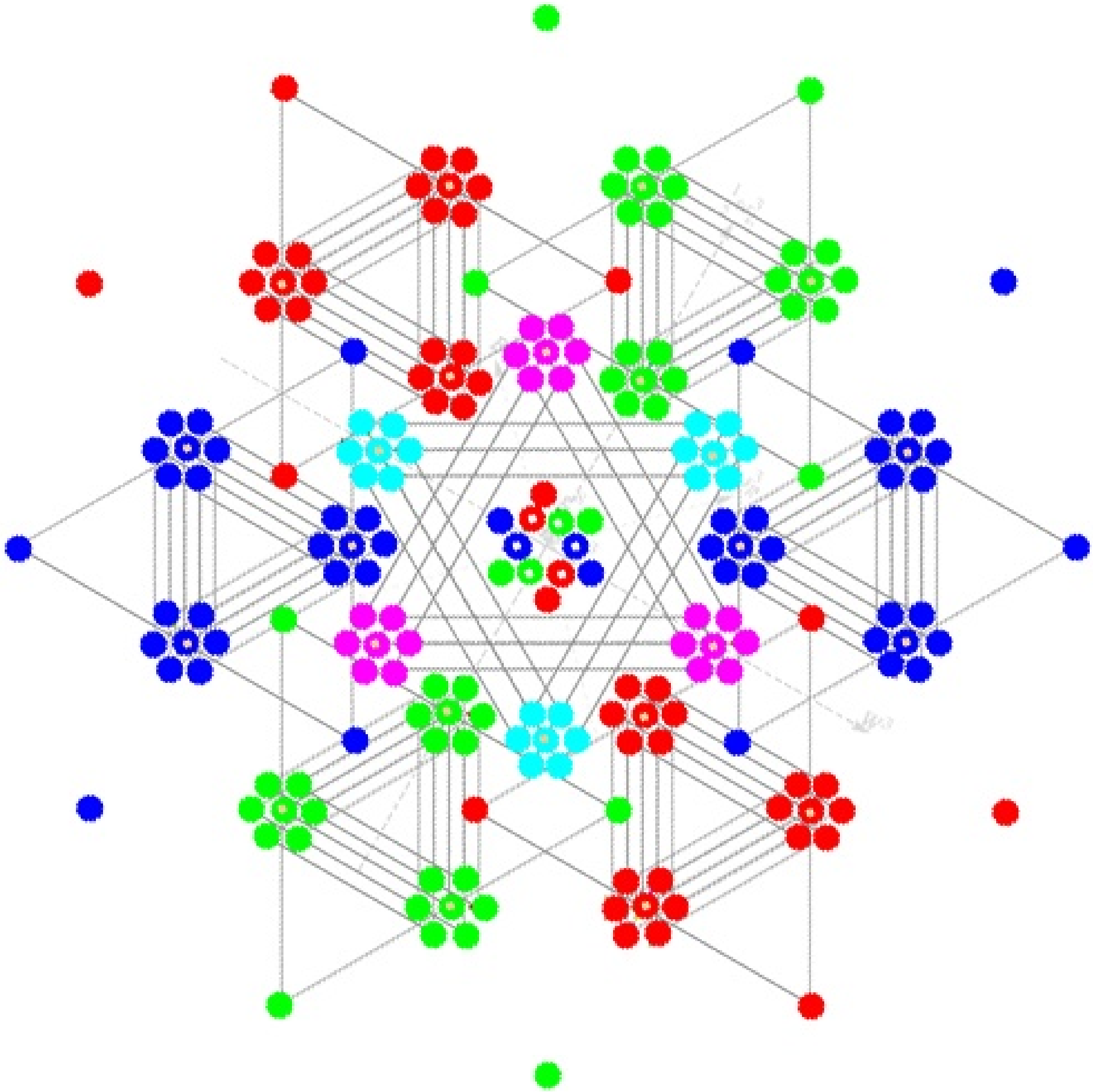
**to get the 72 + 168 = 240 root vectors of the right figure of the above image.**

( Note that 168 is the order of PSL(2,7) = PSL(3,2) and is related to the Klein Quartic. )

Those three images are shown on larger scale in the three images immediately below:







# E8 and PSL(2,11)

According to Bulletin (New Series) of the American Mathematical Society, Volume 36, Number 1, January 1999, Pages 75-93

Finite Simple Groups which Projectively Embed in an Exceptional Lie Group are Classified!

by Robert L. Griess Jr. AND A. J. E. Ryba:

"... The finite subgroups of the smallest simple algebraic group PSL(2;C) (up to conjugacy) constitute the famous list: cyclic, dihedral, Alt4, Sym4, Alt5. This list has been associated to geometry, number theory, and Lie theory in several ways. McKay's correspondence between these groups and the Cartan matrices of types A, D and E and his related tensor product observations are provocative. For the exceptional algebraic groups, theories of Kostant, Springer and Serre have called attention to particular finite simple subgroups. A good list of finite subgroups should help us understand the exceptional groups better. ...



TABLE PE. The finite simple groups with a projective embedding in $E_8(\mathbb{C})$

| $z$ | Finite Simple Group | $G_2(\mathbb{C})$ | $F_4(\mathbb{C})$ | $3E_6(\mathbb{C})$ | $2E_7(\mathbb{C})$ | $E_8(\mathbb{C})$ | Reference, Comments |
|---|---|---|---|---|---|---|---|
| 1 | $Alt_5$ | 4 | 13(8) | 15(10) | $y$ | $\geq 31(\geq 19)Z$ | [F1,2,3][GRQ] |
| 2 | | 4 | 12(21) | 18(32) | $y$ | $\geq 103(\geq 88)S$ | [F1,2,3][GRQ] |
| 1 | $Alt_6$ | 0 | $3A_2A_2$ | $y$ | $y$ | $y$ | [CG][GrG2] |
| 2 | | 0 | $4D_3$ | $y$ | $y$ | $y$ | [CG][GrG2] |
| 3 | | $2(1); 3A_2$ | $3A_2$ | $y$ | $y$ | $y$ | [CG][GrG2][GRQ] |
| 6 | | 0 | 0 | $6A_5$ | $y$ | $y$ | [CG][GrG2][GRQ] |
| 1 | $Alt_7$ | 0 | 0 | $6A_5$ | $2^2D_6$ | $y$ | [CG] |
| 2 | | 0 | $4D_3$ | $y$ | $y$ | $y$ | [CG] |
| 3 | | 0 | 0 | $6A_5$ | $y$ | $y$ | [CG][GRQ] |
| 6 | | 0 | 0 | $6A_5$ | $y$ | $y$ | [CG][GRQ] |
| | $Alt_n$ | | | | | | |
| 1 | $n=8,\dots,10$ | 0 | 0 | 0 | $y: n=8$ | $n \leq 10; 3A_8$ | [CG] |
| 2 | $n=8,\dots,17$ | 0 | $n \leq 9$ | $n \leq 11$ | $n \leq 13$ | $n \leq 17; 2D_8$ | [CG][CW98] |
| | $PSL(2,4) \rightarrow$ | | | | | | $\cong Alt_5$ |
| | $PSL(2,5) \rightarrow$ | | | | | | $\cong Alt_5$ |
| 1 | $PSL(2,7)$ | 2 | $y$ | $y$ | $y$ | $\geq 33(\geq 16)S$ | [K][GRQ] |
| 2 | | 0 | $2B_4$ | $y$ | $y$ | $\geq 88(\geq 22)S$ | [K] |
| 1 | $PSL(2,8)$ | $3(1)P$ | $y$ | $y$ | $y$ | $y$ | [CW83][GrG2] |
| | $PSL(2,9) \rightarrow$ | | | | | | $\cong Alt_6$ |
| 1 | $PSL(2,11)$ | 0 | 0 | $5A_4, 2D_5$ | $y$ | $y$ | [CG][CW83][GRQ] |
| 2 | | 0 | 0 | $2D_5$ | $y$ | $y$ | [CG][CW83][GRQ] |
| 1 | $PSL(2,13)$ | $2(1)P$ | $y$ | $\geq 6(\geq 3)$ | $y$ | $y$ | [CW83][CW93][GrG2] |
| 2 | | 0 | 0 | $2(1); 2A_1A_5$ | $y$ | $y$ | [GRQ] |
| 1 | $PSL(2,16)$ | 0 | 0 | 0 | 0 | $2D_8$ | [CG][CW98][GRQ] |
| 1 | $PSL(2,17)$ | 0 | $2(1); 2B_4$ | $y$ | $4A_7$ | $3A_8$ | [CG] |
| 2 | | 0 | 0 | 0 | 0 | $2D_8$ | [CG] |
| 1 | $PSL(2,19)$ | 0 | 0 | $\geq 4(1)P$ | $y$ | $A_8$ | [CG][CW97][GRQ] |
| 2 | | 0 | 0 | 0 | $y$ | $y$ | [S96]; $y: PGL(2,19)$ |
| 1 | $PSL(2,25)$ | 0 | $P$ | $F_4$ | $y$ | $y$ | [CW97] |
| 2 | | 0 | 0 | 0 | ? | $2D_8$ | [CG] |
| 1 | $PSL(2,27)$ | 0 | $\geq 3(1)P$ | $y$ | $y$ | $y$ | [CW97] |
| 2 | | 0 | 0 | 0 | ? | ? | [CW97] |
| 1 | $PSL(2,29)$ | 0 | 0 | 0 | 0 | $2B_7 \leq 2D_8$ | [CG], p. 370[GRQ] |
| 2 | | 0 | 0 | 0 | $yP$ | $y$ | [SP][GRQ] |
| 1 | $PSL(2,31)$ | 0 | 0 | 0 | 0 | $\geq 3(2)P$ | [S96] [GR31]; 3(2) for $PGL(2,31)$ |
| 1 | $PSL(2,32)$ | 0 | 0 | 0 | 0 | $5(1)P$ | [GR31] |
| 2 | $PSL(2,37)$ | 0 | 0 | 0 | $2(1)P$ | 2(1) | [KR][CG],(5.2.10) |
| 1 | $PSL(2,41)$ | 0 | 0 | 0 | 0 | $3(1)P$ | [GR41] |
| 1 | $PSL(2,49)$ | 0 | 0 | 0 | 0 | $2(1)P$ | [GR41] |
| 1 | $PSL(2,61)$ | 0 | 0 | 0 | 0 | $2(1)P$ | [CGL][GRQ] |
| | $PSL(3,2) \rightarrow$ | | | | | | $\cong PSL(2,7)$ |
| 1 | $PSL(3,3)$ | 0 | $yP$ | $y$ | $y$ | $y$ | in $3^3: SL(3,3)$; [Alek] [CG][GrElAb] |
| 2 | $PSL(3,4)$ | 0 | 0 | 0 | $4A_7$ | $y$ | [CG] |
| 4 | | 0 | 0 | 0 | 0 | $8A_7$ | [CG][GRQ] |
| 6 | | 0 | 0 | $6A_5$ | $y$ | $y$ | [CG] |
| 1 | $PSL(3,5)$ | 0 | 0 | 0 | 0 | $yP$ | in $5^3: SL(3,5)$; [Alek][CG][GrElAb] |
| | $PSL(4,2) \rightarrow$ | | | | | | $\cong Alt_8$ |
| 1 | $PSU(3,3)$ | 1 | $y$ | $y$ | $y$ | $y$ | [CW83][GrG2] |
| 1 | $PSU(3,8)$ | 0 | 0 | 0 | $1P$ | 1 | [GRU][GRQ] |
| 1 | $PSU(4,2)$ | 0 | 0 | $6A_5$ | $y$ | $y$ | $\cong \Omega^-(6,2) \cong W'_{E_6}$ [CG][GRQ] |
| 2 | | 0 | $4D_3$ | $y$ | $y$ | $2D_8$ | |
| 6 | $PSU(4,3)$ | 0 | 0 | $6A_5$ | $y$ | $y$ | [CG] |
| 1 | $PSO^+(8,2)$ | 0 | ? | ? | ? | $2^2D_4D_4$ | [CG] |
| 2 | $PSO^+(8,2)$ | 0 | ? | ? | ? | $2^2D_4D_4$ | [CG] |
| $2^2$ | $PSO^+(8,2)$ | 0 | $2^2D_4$ | $y$ | $y$ | $2^2D_4D_4$ | [CG] |
| | $PSp(4,3) \rightarrow$ | | | | | | $\cong PSU(4,2)$ |
| 2 | $PSp(4,5)$ | 0 | 0 | 0 | 0 | $B_6$ | [CG]; Sec. 6[GRQ] |
| 1 | $PSp(6,2)$ | 0 | 0 | 0 | ? | $2^2D_4^2$ | $\cong \Omega(7,2) \cong W'_{E_7}$ [GRQ];[CG] |
| 2 | | 0 | $2^2D_4$ | $y$ | $y$ | $2^2D_4^2$ | [CG] |
| 1 | $Sz(8)$ | 0 | 0 | 0 | 0 | $3(1)P$ | Sec. 6[GR8] |
| | $G_2(2)' \rightarrow$ | | | | | | $\cong PSU(3,3)$ |
| 1 | $G_2(3)$ | 0 | 0 | 0 | 0 | $D_7$ | [CG][GRQ] |
| 1 | ${}^3D_4(2)$ | 0 | $yP$ | $y$ | $y$ | $y$ | [CW97] |
| 1 | ${}^2F_4(2)'$ | 0 | 0 | $yP$ | $y$ | $y$ | [CW97]; $y: {}^2F_4(2)$ |
| 2 | $HJ$ | 0 | 0 | $6A_5$ | $y$ | $y$ | [CG]; see Sec. 4 |
| 1 | $M_{11}$ | 0 | 0 | $D_5$ | $y$ | $y$ | [CG] |
| 2 | $M_{12}$ | 0 | 0 | 0 | $2B_5 \leq 4D_6$ | $y$ | [CG]; see Sec. 4[GRQ] |

..."

According to a 2006 paper in the Journal of Mathematical Chemistry entitled

"The undecakisicosahedral group and a 3-regular carbon network of genus 26"

by Erwin Lijnen, Arnout Ceulemans, Patrick W. Fowler,and Michel Deza:

> "... the special linear group SL(2,p) ... has order p (p^2 - 1) . The group PSL(2,p) is defined as the quotient group of SL(2,p) modulo its centre ... For all prime numbers p at least 5, the centre has only two elements and the corresponding quotient group PSL(2,p) is simple. Of all these prime numbers p however, the numbers p = 5, 7, 11 stand out as they are the only cases in which the group PSL(2,p) acts transitively on sets of p as well as on sets of p +1 elements, a result already known to Galois.
>
> For all other prime values of p the group PSL(2,p) acts transitively on sets of p + 1 elements, but not on sets of p elements ...
>
> Three projective special linear groups PSL(2,p), those with p = 5, 7 and 11, can be seen as p-multiples of tetrahedral, octahedral and icosahedral rotational point groups, respectively.
>
> The first two have already found applications in carbon chemistry and physics,
>
> PSL(2,5) ... is the rotation group of the fullerene C60 and dodecahedrane C20H20 ... PSL(2,5) has 60 elements and is isomorphic to the pure icosahedral rotation group I . It is alternatively called the pentakistetrahedral group 5T as it contains the tetrahedral group as a subgroup of index 5 . This can easily be seen on a regular dodecahedron where the 20 vertices can be divided into five sets of four vertices such that each set of four vertices forms a regular tetrahedron .... The group PSL(2,5) acts transitively on this set of five tetrahedra by the action of one of the fivefold rotations. The group acts also transitively on a six element set as can be seen from the action on the six diagonals of the regular icosahedron connecting opposite points. ... The smallest 3-regular map with rotational symmetry PSL (2,5) (i.e., 5T or I ) is the all-pentagon dodecahedral map ...
>
> PSL(2,7) is the rotation group of the 56-vertex all-heptagon Klein map, an idealisation of the hypothetical genus-3 "plumber's nightmare" allotrope of carbon. ... PSL(2,7) of order 168, which is alternatively called the heptakisoctahedral group 7O as it contains the octahedral group O as a subgroup of index 7. The group can be represented by the regular genus-3 Klein map, named after Felix Klein who investigated its very high symmetry in connection with the theory of multivalued functions ... Using this map it is easy to show the transitive character on a 7-set, as under removal of the sevenfold symmetry elements, the 56 vertices split into seven octahedral structures containing eight vertices. The complete structure of this group and its relevance to some negative-curvature carbon structures was described in previous papers ... the smallest ... 3-regular map ... with the rotational symmetry PSL(2,7) (i.e., 7O) is the all-heptagon Klein map ...
>
> PSL(2,11) ... has potential relevance for the study of the icosahedral phase of quasicrystals, and was identified as a finite simple subgroup of the Cartan exceptional group E8 ... Here, we present an analysis of PSL(2,11) as the rotation group of a 220-vertex, all 11-gon, 3-regular map, which provides the basis for a more exotic hypothetical sp2 framework of genus 26. The group structure and character table of PSL (2,11) are developed in chemical notation and a three dimensional (3D) geometrical realisation of the 220-vertex map is derived in terms of a punctured polyhedron model where each of 12 pentagons of the truncated icosahedron is connected by a tunnel to an interior void and the 20 hexagons are connected

tetrahedrally in sets of 4. ... to realise PSL(2,11) (i.e., 11 I ) by a 3-regular map it is necessary to go to an all-undecagon map which will have 220 vertices, v, and 330 edges, e and 60 faces, f . Hence, from f = v / 2 + 2( 1 - g ), we find a genus g of 26. ... the map of interest ...[has]... total automorphism group consists of 1,320 elements, of which the orientation-preserving (rotational) part of 660 elements corresponds with the group PSL(2,11). ... a geometrical representation for this genus-26 map has thus far not yet been reported. The most obvious representation would be to draw a Schlegel-like diagram consisting of a central 11-gon surrounded by layers of undecagonal faces, adding layers until all faces have been accounted for. ...

*E.Lijnen et al. / The undecakisicosahedral group*

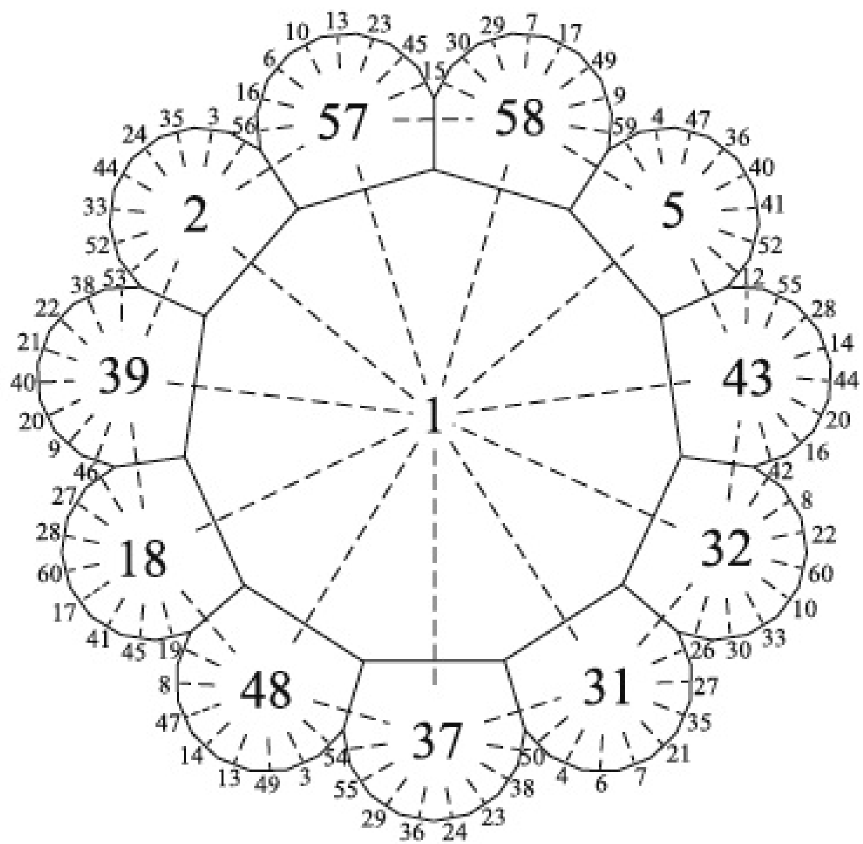



Figure 1. Partial Schlegel-like diagram representing the 220-vertex regular map of genus 26. The face-numbering corresponds with the numbering used in table 1 to describe the 60-vertex dual map. The edges of the dual map are drawn as dashed lines.

... Continuation to produce the whole diagram with 220 numbered vertices and all 60 faces would yield a very intricate figure. Instead, we work with the dual map, represented by the dashed lines in figure 1. It consists of 60 undecavalent vertices and 220 triangular faces, and of course retains the PSL(2,11) rotational symmetry of the original 220-vertex 3-regular map. ...

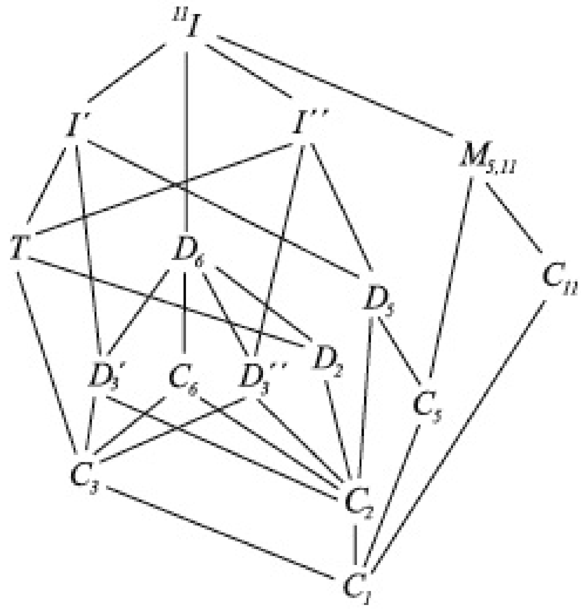



Figure 2. Genealogical tree of subgroups of the undecakisicosahedral group $^{11}I$.

... our parent group has four direct subgroups: I' , I" , M5,11 and D6. In total there are 22 subgroups isomorphic to the purely rotational icosahedral group. They fall into two subgroup classes I' and I" , which are non-equivalent within 11I symmetry. The subgroups within one of these classes are transformed into each other by any one of the 11-fold operations. Note, that equivalence of both classes is restored when one considers the full symmetry group 11Id of the genus-26 map, which also includes orientation-reversing symmetry operations. The second largest subgroup class consists of 12 groups of order 55 corresponding with the metacyclic group M5,11, which is formed by the semi-direct product of a fivefold and 11-fold cyclic group and is the only subgroup of 11I that is not isomorphic with a point group. The fourth direct subgroup class contains 55 groups of order 12 isomorphic to a sixfold dihedral group. Apart from the subgroup class T with 55 purely rotational tetrahedral groups, all other subgroup classes are only composed of dihedral groups Dn or cyclic groups Cn. ...

We ... investigate the possibility of forming a 3D geometrical model exhibiting such icosahedral symmetry, where we further impose the restriction that the 60 vertices remain equivalent, as is the case under PSL(2,11) symmetry. In 3D space there are four semiregular convex polyhedra on 60 vertices obeying these restrictions. They are the four icosahedral Archimedean solids on 60 vertices depicted ...

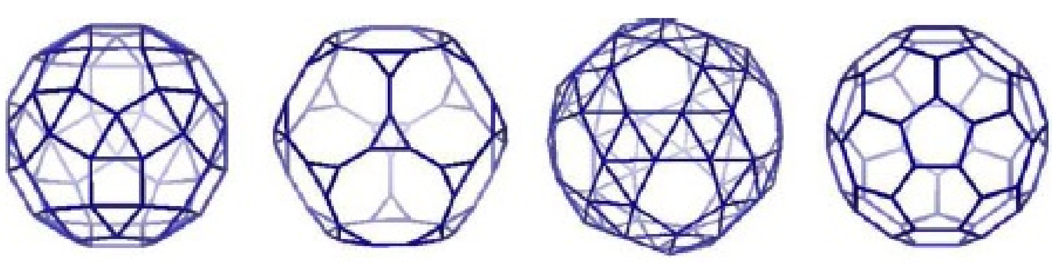


Figure 3. The four icosahedral Archimedean solids on 60 vertices. From left to right: the small rhombicosidodecahedron, the truncated dodecahedron, the snub dodecahedron and the truncated icosahedron.

... namely the small rhombicosidodecahedron, the truncated dodecahedron, the snub dodecahedron and the truncated icosahedron. ... Seeking a geometrical representation it is worth investigating whether the

> graphs of these Archimedean solids appear as subgraphs of the graph underlying our 26-genus map. If such an Archimedean subgraph does indeed exist, i would be useful as a 3D icosahedral backbone on which a complete geometrical model of the genus-26 map could be built. ...
>
> The ... most interesting subgraph is the truncated icosahedron, corresponding with the framework of Buckminsterfullerene C60. The special relationship of this truncated icosahedral structure to the group PSL(2,11) has already been noted in papers by Kostant ...[who]... showed that the graph of C60 can be expressed group-theoretically by the structure of a 60-element conjugacy class of PSL(2,11) ...".

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# E8(p)

According to "The Classification of the Finite Simple Groups" (AMS Mathematical Surveys and Monographs, Vol. 40, No. 1, 1994) by Gorenstein, Lyons, and Solomon ( in the following I change their notation from prime number q to prime number p ):

> "... It is our purpose ... to prove the following theorem:
>
> CLASSIFICATION THEOREM. Every finite simple group is
>
> - cyclic of prime order,
> - an alternating group,
> - a finite simple group of Lie type,
> - or one of the twenty-six sporadic finite groups.
>
> ... the bulk of the set of finite simple groups consists of finite analogues of Lie groups ... called finite simple groups of Lie type, and naturally form 16 infinite families ... In 1968, Steinberg gave a uniform construction and characterization of all the finite groups of Lie type as groups of fixed points of endomorphisms of linear algebraic groups over the algebraic closure of a finite field ...
>
> The finite simple groups are listed ...[including]... Group ... E8(p) ...[ for prime p ]...
>
> Order ... p^120 (p^2 - 1 ) (p^8 - 1 ) (p^12 - 1 ) (p^14 - 1 ) (p^18 - 1 ) (p^20 - 1 ) (p^24 - 1 ) (p^30 - 1 ) ...".

To get a feel for E8(p), ignore the -1 part of the Order formula for E8(q) and see that the order of E8(q) is roughly (somewhat less than) p^120 p^(2+8+12+14+18+20+24+30) = p^(120+128) = p^248

Note that 248-dim E8 = 120-dim adjoint of Spin(16) + 128-dim half-spinor of Spin(16)

and that p^248 is the set of maps from 248 to p

and that the exponents are one greater than each of the primes 1, 7, 11, 13, 17, 19, 23, and 29,

but not similarly related to the primes to 2, 3, or 5.

and that

- E8(2) = the number of ways to assign the 2 elements + and 1 (as in + and - electric charge of the U(2) electroweak gauge group) to each of the 248 basis elements of E8
- E8(3) = the number of ways to assign the 3 = 2+1 = 4-1 elements + and 1 (as in r, g and b color charge of the SU (3) color force gauge group) to each of the 248 basis elements of E8
- E8(5) = the number of ways to assign the 5 = 6-1 = 4+1 elements x, y, z, t and m (as in spatial x, y and z , and time t and scale/mass m of the Spin(2,3) anti-deSitter group of MacDowell-Mansouri gravity) to each of the 248 basis elements of E8
-
- E8(7) = the number of ways to assign the 7 = 6+1 = 8-1 Imaginary Octonion basis elements (as in spatial/ internal symmetry part of 8-dim Kaluza-Klein spacetime and tree-level-massive first generation fermion particles and antiparticles and in 7 of the 8 Dirac gammas of E8 physics) to each of the 248 basis elements of E8
- E8(11) = the number of ways to assign 11 = 12-1 elements (as in the 11 generators of charge-carrying SU(3) and SU(2) of the 12 generators of the Standard Model SU(3)xSU(2)xU(1) in E8 physics) to each of the 248 basis elements of E8
- E8(13) = the number of ways to assign 13 = 12+1 elements (as in 12 root vectors of Conformal Spin(2,4) = SU (2,2) of MacDowell-Mansouri gravity in E8 physics) to each of the 248 basis elements of E8
- E8(17) = the number of ways to assign 17 = 16+1 elements as in the 16-dim vector representation of Spin(16) and the 16-dim full spinor representation of Spin(8) and 16-dim pairs of octoniions representing second-generation fermions and in the complexification of 8-dim Kaluza-Klein spacetime and 8-dim representation spaces of first-generation particles and antiparticles) to each of the 248 basis elements of E8
- E8(19) = the number of ways to assign 19 = 18+1 elements (as in the 18 root vectors of 21-dimensional rank 3 Spin(7)) to each of the 248 basis elements of E8
- E8(23) = the number of ways to assign 23 = 24-1 elements (as in 24-dim triples of octonions representing third-generation fermions and 24 full octonionic dimensions of the 27-dim Jordan algebra J(3,O)) to each of the 248 basis elements of E8
- E8(29) = the number of ways to assign 29 = 28+1 elements (as in 28-dim d4 for MacDowell-Mansouri gravity and 28-dim d4 for the Standard Model in E8 physics) to each of the 248 basis elements of E8
- ...
- E8(113) = the number of ways to assign 113 = 112+1 elements (as in the 112 root vectors of 120-dim Spin(16)) to each of the 248 basis elements of E8
- E8(127) = the number of ways to assign 127 = 128-1 elements (as in 64+64 = 128-dim half-spinors of Spin(16) representing first-generation fermion particles and antiparticles, and the related Dirac Gammas) to each of the 248 basis elements of E8
- ...
- E8(257) = the number of ways to assign 257 = 256+1 elements (as in 256-dim Cl(8)) to each of the 248 basis elements of E8
- ...

- E8(65537) = the number of ways to assign 65,537 = 65,536+1 elements (as in 65,536-dim Cl(16)) to each of the 248 basis elements of E8

In math.RT/0712.3764 Skip Garibaldi said:

> "... Theorem. Let L be a Lie algebra of type E8 over a field of characteristic 5. Then there is no quotient trace form on L. ...
>
> Roughly speaking, we use lemmas due to Block to reduce to showing that the trace is zero for representations coming from algebraic groups of type E8. From this, it is easy to see that it suffices to consider only the Weyl modules, which are defined over Z. Leaning on the fact that a Lie algebra of type E8 is simple over every field ... we note that the trace form is zero because 5 divides 60, the Dynkin index of E8. ...
>
> Lemma 1.3. Let G and g be ... of type E8. The following are equivalent:
>
> - (1) The Killing form of g is not zero over F.
> - (2) The Killing form of g is nondegenerate over F.
> - (3) The characteristic of F is =/= 2, 3, 5.
>
> ...
>
> Proposition 1.5. Let G and g be as in 1.1 and of type E8. There is a representation rho of G over F with tr =/= 0 if and only if F has characteristic =/= 2, 3, 5.
>
> ... ".