Tony Smith's Home Page

Primitive Idempotents for Cl(8) Clifford Algebra

Ian Porteous, in Lecture 2: Mathematical Structure of Clifford Algebras, presented at "Lecture Series on Clifford Algebras and their Applications", May 18 and 19, 2002, as part of the 6th International Conference on Clifford Algebras and their Applications in Mathematical Physics, Cookeville, TN, May 20-25, 2002, said:

> "... the Clifford algebra ... Cl(p,q) ...[is]... the real algebra of endomorphisms of a right A-linear space of the form A^m, where A = R, C, H, 2^R or 2^H. This space is called the (real) **spinor** space or space of (real) spinors of the orthogonal space R(p,q). It is identifiable with a minimal left ideal of the algebra, namely the space of matrices with ever column except the first non-zero. However as a minimal left ideal it is non-unique. ...
>
> **Minimal left ideals of a matrix algebra are generated by primitive idempotents**. An idempotent of an algebra is an element y such that y^2 = y. It is primitive if it cannot be expressed as the sum of two idempotents, whose product is zero. The simplest example in a matrix algebra is **the matrix consisting entirely of zeros, except for a single entry of 1 somewhere in the main diagonal**. The minimal ideal generated by such an idempotent then consists of matrices all of whose columns are zero except one consist of zeros. The easiest idempotents to construct are of the form (1/2)(1 + x) where x^2 = 1, but not x^2 = -1. Then of course (1/2)(1 - x) is also an idempotent, so that spinor spaces constructed in this way come naturally in pairs. However these are not necessarily primitive. They are where the matix algebra consists of 2x2 matrices over R, C or H, but in the case of 4x4 matrix algebras the primitive idempotents are products of commuting pairs of such idempotents.
>
> Though as minimal left ideals of matrix algebras any two spinor spaces are equivalent, they may lie differently when the Clifford algebra structure of the matrix algebra is taken into account ...".

Bilge, Dereli, and Kocak, in their paper "The geometry of self-dual two-forms", J. Math. Phys. 38 (1997) 4804-4814, say in their abstract:

> "... We show that self-dual two-forms in 2n-dimensional spaces determine a n^2 &endash;

n + 1 -dimensional manifold S_2n and the dimension of the maximal linear subspaces of S_2n is equal to

the (Radon&endash;Hurwitz) number of linearly independent vector fields on the sphere S^(2n &endash; 1).

We provide a direct proof that for n odd S_2n has only one-dimensional linear submanifolds. We exhibit 2^c &endash; 1-dimensional subspaces in dimensions which are multiples of 2^c, for c=1,2,3. In particular, we demonstrate that **the seven-dimensional linear subspaces of S_8 also include among many other interesting classes of self-dual two-forms**, the self-dual two-forms of Corrigan, Devchand, Fairlie, and Nuyts [Nucl. Phys. B 214, 452 (1983)] and a representation of Cl_7 given by octonionic multiplication. We discuss the relation of the linear subspaces with the representations of Clifford algebras. ...".

Pertti Lounesto, in his book Clifford Algebras and Spinors (Second Edition, LMS 286, Cambridge 2001) says at pages 226-227 and 29:

"... **Primitive idempotents and minimal left ideals**

An orthonormal basis of of R(p,q) induces a basis of Cl(p,q), called the standard basis. Take a non-scalar element e_T, e_T^2 = 1, from the standard basis of Cl(p,q). Set e = (1/2)(1 + e_T) and f = (1/2)(1 - e_T), then e + f = 1 and ef = fe = 0. So Cl(p,q) decomposes into a sum of two left ideals

Cl(p,q) =Cl(p,q) e + Cl(p,q) f ,

where dim Cl(p,q) e = dim Cl(p,q) f = [dim] (1/2) Cl(p,q) = 2^(n-1) [for n = p + q]. Furthermore, if { e_T_1 , e_T_2 , ... , e_T_k } is a set of non-scalar basis elements such that e_T_i^2 = 1 and e_T_i e_T_j = e_T_j e_T_i , then **letting the signs vary independently in the product**

(1/2)(1 +/- e_T _1) (1/2)(1 +/- e_T_2) ... (1/2)(1 +/- e_T_k) ,

one obtains 2^k idempotents which are mutually annihilating and sum up to 1. The Clifford algebra Cl(p,q) is thus decomposed into a direct sum of 2^k left ideals, and by construction, **each left ideal has dimension 2^(n - k)** . In this way one obtains a minimal left ideal by forming a maximal product of non-annilating and commuting idempotents.

The Radon-Hurwitz number r_i for i in Z is given by

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

and the recursion formula r_(i + 8) = r_ i + 4 . For the negative values of i one may observe that r_(-1) = -1 and r_(-i) = 1 - i + r_(i + 2) for i > 1 .

Theorem. In the standard basis of Cl(p,q) there are always k = q - r_(q - p) non-scalar elements e_T_i , e_T_i^2 = 1 , which commute, e_T_i e_T_j = e_T_j e_T_i , and generate a group of order 2^k . The product of the corresponding mutually non-annihilating idempotents,

f = (1/2)(1 +/- e_T _1) (1/2)(1 +/- e_T_2) ... (1/2)(1 +/- e_T_k) ,

is primitive in Cl(p,q). Thus, the left ideal S = Cl(p,q) f is minimal in Cl(p,q).

Example ... In the case of R(0,7) we have k = 7 - r_7 = 4. Therefore the idempotent

f = (1/2)(1 + e_124) (1/2)(1 + e_235) (1/2)(1 + e_346) (1/2)(1 + e_457)

is primitive to Cl(0,7) = 2^Mat(8,R). ...

... If e and f are commuting idempotents of a ring R, then e f and e + f - e f are also idempotents of R. The idempotents e f and e + f - e f are a greatest lower bound and a least upper bound relative to the partial ordering given by

e ≤ f if and only if ef = fe = e

A set of commuting idempotents induces a lattice of idempotents. ...

...

... The Clifford algebra ... has three involutions similar to complex conjugation in C. ... The grade involution is an automorphism ... while the reversion and the Clifford-conjugation are anti-automorphisms ...".

Pertti Lounesto, in his book Spinor Valued Regular Functions in Hypercomplex Analysis (Report-HTKK-MAT-A154 (1979) Helsinki University of Technology) says [in the quote below I have changed his notation for a Clifford algebra from R_(p,q) to Cl(p,q)] at pages 40-42:

"... To fix a minimal left ideal V of Cl(p,q) we can choose a primitive idempotent f of Cl(p,q) so that V = Cl(p,q) f . By means of an orthonormal basis { e_1 , e_2 , ... , e_n } for [the grade-1 vector part of Cl(p,q)] Cl^1(p,q) we can construct a primitive idempotent f as follows: Recall that the 2^n elements

e_A = e_a_1 e_a_2 ... e_a_k , 1 ≤ a_1 < a_2 < ... < s_k ≤ n

constitute a basis for Cl(p,q). ... dim_R V = 2^X , where X = h or X = h + 1 according as p - q = 0, 1, 2 mod 8 or p - q = 3, 4, 5, 6, 7 mod 8 and h = [n / 2] . Select n - X elements e_A, e_A^2 = 1 , so they are pairwise commuting and generate a group of order 2^(n - X) . then the idempotent ...

f = (1/2)(1 + e_A_1) (1/2)(1 + e_A_2) ... (1/2)(1 + e_A_(n - X))

is primitive ... To prove this note that the dimension of (1/2)(1 + e_A) Cl(p,q) is (2^n) / 2 and so the dimension of Cl(p,q) f is (2^n) / (2^(n - X)) = 2^X . Hence, if there exists such an idempotent f , then f is primitive. To porove that such an idempotent f exists in every Clifford algebra Cl(p,q) we may first check the lower dimensional cases and then proceed by making use of the isomorphism Cl(p,q) x Cl(0,8) = Cl(p, q + 8) and the fact that **Cl(0,8) has a primitive idempotent**

f = (1/2)(1 + e_1248) (1/2)(1 + e_2358) (1/2)(1 + e_3468) (1/2)(1 + e_4578) =

=(1/16)(1 + e_1248 + e_2358 + e_3468 + e_4578 + e_5618 + e_6728 + e_7138 - e_3567 - e_4671 - e_5712 - e_6123 - e_7234 - e_1345 - e_2456 + e_J)

with four factors [and where J = 12345678] ...

The division ring F = f Cl(p,q) f = { PSI in V | PSI f = f PSI } is isomorphic to R, C, or H according as p - q = 0, 1, 2, mod 8, p - q = 3 mod 4, or p - q = 4, 5, 6 mod 8. The map ...

V x F -> V , (PSI, /\ ) -> PSI /\

defines a right F-linear structure on V. Provided with this right F-linear structure the minimal left ideal V of Cl(p,q) will be called the pinor module. Similarly, beginning with a minimal left ideal of the even subalgebra Cl(p,q)^(0) we obtain the **spinor** module. ...

... take a left ideal W = V or W = V + V' , where V' = { PSI' | PSI in V } , according as Cl(p,q) is simple or a direct sum of two simple ideals (1/2) (1 +/- e_J) Cl(p,q) . Take an idempotent e = f or e = f + f' , respectively.

The ring E = e Cl(p,q) e = { PSI in W | PSI e = e PSI } is E = F or E = F + F', according as Cl(p,q) is simple or a direct sum of two simple ideals (1/2) (1 +/- e_J) Cl(p,q) . The ring E is isomorphic to R, C, H, 2^H, or 2^R . The map ...

W x E -> W , (PSI, /\ ) -> PSI /\

defines a right E-linear structure on W. Provided with this right E-linear structure the left ideal W of Cl(p,q) will be called the binor module.

Let B be either of the anti-involutions B+ or B- of Cl(p,q) . The real linear spaces

P+ = { PSI in V | B(PSI) = +PSI }

P- = { PSI in V | B(PSI) = -PSI }

have dimensions 0, 1, 2, or 3 and

P = P+ + P- = { PSI in V | B(PSI) in V }

has dimension 0, 1, 2, 3, or 4 no matter how large is the dimension of V. To prove this we may use the facts that Cl(p,q) x Cl(0,8) = Cl(p, q + 8) and for Cl(0,8) the real dimension of P = P+ is 1. The real linear space B = { PSI in W | B(PSI) in W } has dimension 1, 2, 4, or 8. ...".

To paraphrase Pertti Lounesto:

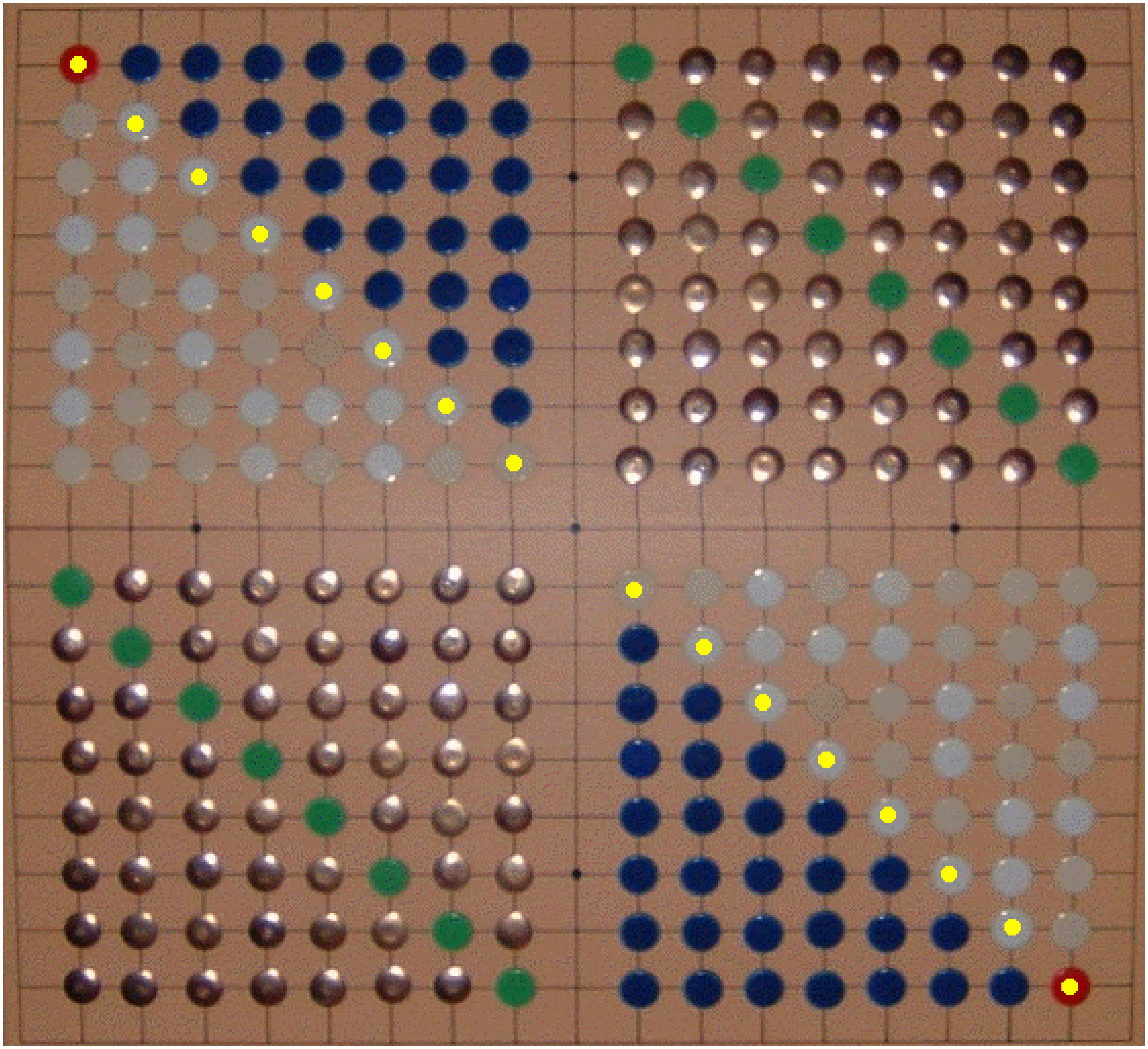
In the case of Cl(0,8) we have k = 8 - r_8 = 8 - 4 = 4. Therefore **Cl(0,8) has a primitive idempotent**

f = (1/2)(1 + e_{1248}) (1/2)(1 + e_{2358}) (1/2)(1 + e_{3468}) (1/2)(1 + e_{4578})

By letting the signs vary independently in the product we get a set of 2^4 = 16 idempotents.

Consider the graded structure of the 256 elements of Cl(0,8) = Cl(1,7). In the image below, there are:

- 1 + 1 = 2 red scalar and its dual;
- 8 + 8 = 16 green vectors and their duals;
- 28 + 28 = 56 blue bivectors and their duals;
- 56+56 = 112 gold trivectors and their duals; and
- 35 + 35 = 70 white middle-grade 4-vectors, the set of which is self-dual.



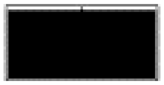
The 1 + 7 + 7 + 1 = 16 diagonal elements (marked in yellow - 2 scalars and 14 4-vectors) correspond to the 16 terms in the primitive idempotent

f = (1/2)(1 + e_1248) (1/2)(1 + e_2358) (1/2)(1 + e_3468) (1/2)(1 + e_4578) =

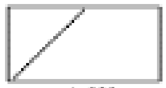
=(1/16)(1 + e_1248 + e_2358 + e_3468 + e_4578 + e_5618 + e_6728 + e_7138 - e_3567 - e_4671 - e_5712 - e_6123 - e_7234 - e_1345 - e_2456 + e_J)

Note that the 16 terms in the primitive idempotent correspond to 16 of Wolfram's 256 Cellular Automata:

- + e_12345678


rule 255
11111111

- + e_6728 + e_3468 + e_4578 to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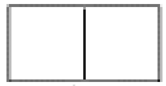
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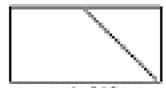
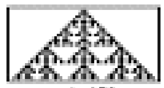

rule 216
11011000

- + e_2358 + e_1248 + e_5618 + e_7138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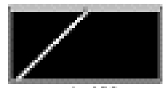
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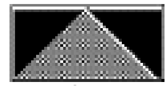
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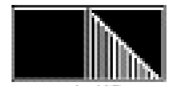
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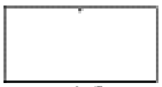
- + 1 to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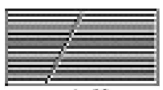
- - e_5712 - e_1345 - e_6123


rule 83
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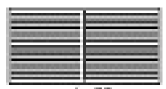
rule 29
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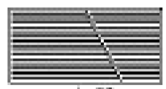
rule 39
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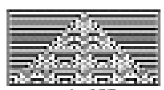
- - e_4671 - e_7234 - e_2456 - e_3567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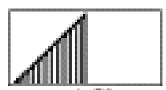
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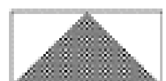
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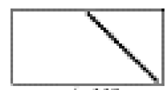
rule 116
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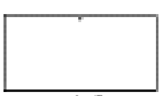
Note the Cl(0,8) = Cl(1,7) triality correspondences among:

- the 8 +half-spinors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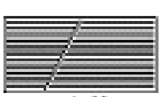
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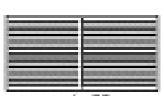
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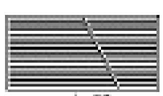
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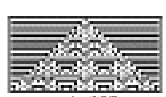
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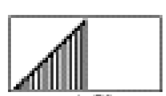
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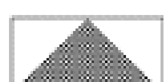
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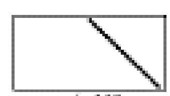
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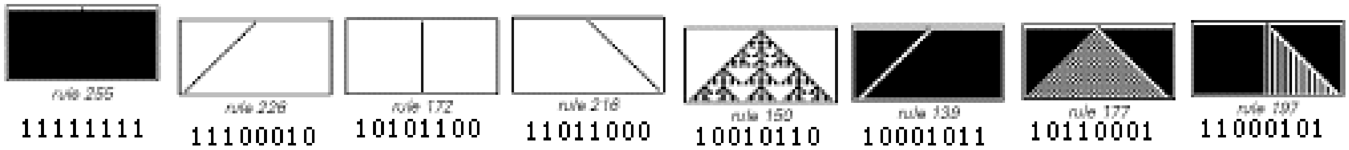
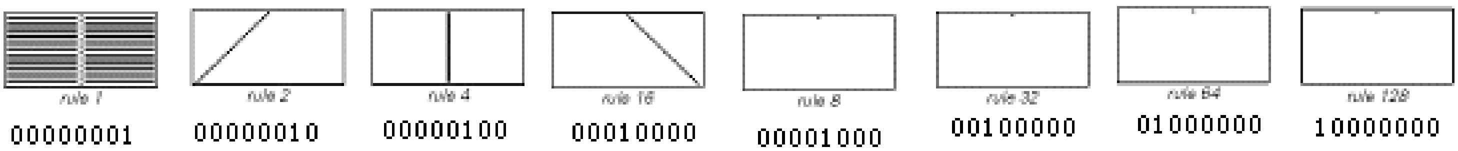
- the 8 -half-spinors



- the 8 vectors



With respect to Cellular Automata:

Michael Gibbs has been working on using Cellular Automata as neural network nodes, and Robert de Marrais has written a Box-Kites III paper (at math/0403113), leading me to think of some questions:

- Could 16x16 structures such as switching yards of Box-Kites III have structures corresponding to the graded structure o the Clifford algebra Cl(8) that is the 16x16 real matrix algebra?
- Since the vectors of the Cl(8) Clifford algebra are 8-dimensional and correspond to the octonions, if you take the correspondences between the 256 Wolfram CA and the Cl(8) basis elements described here, there is a correspondence:

CA Rule No.	Octonion Basis Element
1	1
2	i
4	j
16	k
8	E
32	I
64	J
128	K

- Could such a correspondence be used to construct such things as "Box-Kites" whose vertices might be regared, not just as octonions etc, but also as Cellular Automata?
- Could Box-Kite type structures give useful computational structures if the vertices were considered as CA and the edge-flow-orinetations were considered as information flow in a computing system?
- If such a computing system can be set up for 2^n-ionic structures for large n, then, since for 16-ions and larger you have interesting zero-divisor "sleeper-cell" substructures, could they be useful

with respect to computational systems, perhaps doing things like forming loops that might let the computational system to "adjust itself" and/or "teach itself"?

Robert de Marrais commented on some of those questions, saying in part:

"... I'm finding two directions to go with box-kites next, and yes, cellular automata clearly are part of it. I did a poster session at Wolfram's [2004 NKS] conference, had a long talk with him and another fellow, one Rodrigo Obando ... My poster session is currently being written up for incorporation in the conference proceedings, after which it goes out to arXiv.org -- and will have (or so I hope!) some nifty graphics for higher-dimensional cases. ... But now to the two directions, which relate to your suggestions:

(1) Boolean monotone and antitone function-pairings can be used, per Rodrigo Obando, to generate exactly all and only the complex cellular automata for a given n and r . . . and, given that for n=4 that means Dedekind's number of 168 mono- and iso- tone functions each, connections to box-kites immediately suggest themselves ... He tells me his work is leading him not merely to isolate and catalog the "complex" CA's for high n and r, but that he's finding -- when he generalizes to the n => infinity situation, that he gets violations of the continuum hypothesis ...

(2): spin networks. The key revelation (which I telescoped on the last couple pages of "Box Kites III") concerns what I call the "trip-sync property." As it turns out, this is incredibly easy to prove, for all box-kites in all dimensions. ... what is truly interesting is this: zero-divisor systems are, ironically, PRESERVERS of associative order! Specifically, each of the four "sails" on a box-kite can be represented (on an isomorphic box-kite diagram, in fact!) as a system of four interconnected Quaternion copies: write each vertex as a pairing of one uppercase and one lowercase letter (with the 'generator' of the given 2^n-ions being the divider of the two: e.g., with the Sedenions, g = the index-8 imaginary, and the pure Sedenions of index > 8 are "uppercase," with the Octonions thereby being written with "lowercase" letters). Using the standard notation in my "strut tables," the "triple-zigzag sail" has vertices (A + a), (B + b), (C + c). Since it's a triple zigzag, this means all the edge-signs are negative: hence, if one takes the diagonal "/" in the (A, a) plane, it will zero-divide the diagonal slanting like "\" in either the (B, b) or (C, c) planes. Now consider that there are 4 associative triplets here: (a, b, c); (a, B, C); (A, b, C); and, (A, B, c). Now, allow for "slippage" of the following sort: orbitings among (a, b, c) can be imagined to "slip" into one of the other 3 by keeping one of the lowercases unchanged, but allowing the other two to form "resonances" with the generator (the XOR of two uppercase is, of course, a lowercase). The trip-sync property says this: IF the "sail" is the triple zigzag, all such slippage can occur without any "flips" in orientation; however, IF the "sail" is one of the other three "trefoil" sails, then ONLY slippage with the lowercase being one of the triple zigzag's trio will preserve orientation. Importantly, this gives a way to envision "observable" and "unobservable" in a quantum mechanical manner: orientation REVERSAL will be observable, and the isomorphism of quaternion algebra to SU2 gives you (recall my graphics toward the end of the first Box-Kite paper vis a vis Catastrophe Theory?) two orthogonal circles whose centers are the "units" of two lines of diagonal idempotents (which, like the diagonals in the boxkite vertex-planes, are ALSO zero-dividers -- but only with each other!!). That is, the 4 axes in the SU2 representation are reals, the usual imaginaries, Pauli spin-matrix "mirror numbers" which square to +1, and a "commutative i" which commutes between these latter two. (This is both Cl(2) in Clifford algebra lingo, and Muses' simplest epsilon-number space.) But then, the centers of the two orthogonal circles are just the projection operators -- 1/2(1 +/- m), m the Pauli "mirror axis" unit. As systems of box-kites get very entangled in higher dimensions (in 32-D, you have systems of 7 of them forming what I call Pleiades, with some fascinating synergetic properties), spin-foams with self-organizing potential suggest themselves . . .

Now, (1) and (2) are BOTH related to my ultimate objective, which is not physics per se, but rather Levi-Strauss's canonical law of myths, and the creation of an infinite-dimensional "collage space" that can accommodate his systems of mythopoetic sign-shunting in a manner roughly reminiscent of Fourier series' infinite-dimensional backdrop for generalized harmonics. So that means I'll be busy with my hobbyhorse at least through "Box-Kites VI"!

The key notion here is that each sail can be seen as an ensemble of 5 Quaternion copies (the 4 associative triplets each are completed by the real unit, and the "sterile" zero-divisor-free triplet of generator, strut constant, and their XOR makes 5). Viewing things in closest-packing-pattern style, we have 5 interacting "unit quaternion" algebras -- with the interactions entailing (1, u), where 'u' is the shared non-real unit. Interestingly, this gives a nice way to think about the Tibetan Book of the Dead's "58 angry demons and 42 happy Buddhas," 100 in all = 5 * 16 + 2*10 = 100 distinct units in the interlinked 5-fold "unit quaternion" ensemble. So one first sees the "42 Assessors," then zooms in one one of the 7 isomorphic box-kites (which, as with all isomorphies, can be seen as identical at some higher level); then, one zooms in further on the "second box-kite" which has its struts defined by upper vs. lower case letters, and the triple zigzag analog being the "all lowercase" sail. ... I've also just purchased a domain name -- "TheoryOfZero.com" -- where I'll start building a site as soon as time permits. ... All these threads are getting ever more entangled and intriguing, aren't they? ...".

Tony Smith's Home Page

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