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$SU(3) \times SU(2) \times U(1)$, Higgs, and Gravity from $Spin(0,8)$ Clifford Algebra $Cl(0,8)$

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Abstract

The Lagrangian action for the $D_4 - D_5 - E_6$ model of hep-th/9306011 [12]

$$\int_{V_8} F_8 \wedge \star F_8 + \overline{S_8^+} \not\partial S_8^- + GF + GH$$

has 8-dim spacetime V_8 of the vector representation of $Spin(0,8)$
8-dim fermion fields $S_8^+ = S_8^-$ of the half-spinor reps of $Spin(0,8)$ and
28 gauge boson fields F_8 of the bivector adjoint rep of $Spin(0,8)$.

In this paper, the structure of the positive definite Clifford algebra $Cl(0,8)$ of $Spin(0,8)$, and the triality automorphism $V_8 = S_8^+ = S_8^-$, are used to reduce the spacetime to 4 dimensions and thereby change the gauge group from $Spin(0,8)$ to the realistic $SU(3) \times SU(2) \times U(1)$, Higgs, and Gravity.

The effect of dimensional reduction on fermions,
to introduce 3 generations, has been described in hep-ph/9301210 [13].

The global geometry of manifolds $V_8 = S_8^+ = S_8^- = \mathbf{R}P^1 \times S^7$, the effects of dimensional reduction on them, and the calculation of force strength constants, has been described in hep-th/9302030 [14].

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1 Introduction

The Lagrangian action for the $D_4 - D_5 - E_6$ model of hep-th/9306011 [12]

$$\int_{V_8} F_8 \wedge \star F_8 + \overline{S_8^+} \not{\partial} S_8^- + GF + GH$$

has 8-dim spacetime V_8 of the vector representation of $Spin(0,8)$
8-dim fermion fields $S_8^+ = S_8^-$ of the half-spinor reps of $Spin(0,8)$ and
28 gauge boson fields F_8 of the bivector adjoint rep of $Spin(0,8)$.

In this paper, the structure of the positive definite Clifford algebra $Cl(0,8)$ of $Spin(0,8)$, and the triality automorphism $V_8 = S_8^+ = S_8^-$, are used to reduce the spacetime to 4 dimensions and thereby change the gauge group from $Spin(0,8)$ to the realistic $SU(3) \times SU(2) \times U(1)$, Higgs, and Gravity.

This paper is an extension of a series of papers [12, 13, 14, 15] attempting to construct a realistic model of particle physics and gravity using structures related to Spin(0,8), the unique Lie algebra with a triality automorphism between its vector representation and each of its two half-spinor representations. All three of those representations are 8-dimensional, and octonionic structures seem to be at the heart of the special structures used in building the model.

Similar octonionic structures have appeared in other areas of mathematics and physics, and some of the people working with them attended a meeting in January 1994 at the Institute for Theoretical Physics at Chalmers University, Göteborg, Sweden, hosted by Martin Cederwall and organized by Geoffrey Dixon. Octonionic structures in superstrings were presented by Martin Cederwall and Corinne Manogue; Division algebra structures were presented by Geoffrey Dixon; and Clifford algebra structures were presented by Rafal Ablamowicz, Pertti Lounesto, and Ian Porteous. Just prior to the Chalmers meeting, my thinking was influenced by discussions in Lübeck with Wolfgang Mantke.

This paper is my attempt to use their ideas to further the construction of the Spin(0,8) model. To the extent that this paper is useful, credit should go to them. However, they should not be blamed for wrong or useless material in this paper.

An outline of the path from $Spin(0,8)$ to $SU(3) \times SU(2) \times U(1)$, Higgs, and Gravity is the following table of contents:

Some references to material used in this paper are:

The general structure of Clifford algebras is described in the book of Ian Porteous [11]. A new third edition of his book should be out later this year. His book not only describes Clifford algebras, but is also a good introduction to division algebras and the geometric actions of *Spin* groups on spheres.

The papers of Geoffrey Dixon [4] are particularly useful references to division algebras, including the use of division algebras with respect to spinor spaces of *Spin* groups.

The global structure of *Spin* groups is well covered in the paper of Corinne Manogue and Jörge Schray [8].

The torsion structure of S^7 and octonions, leading to a generalized S^7 algebra, is given by Martin Cederwall and Christian Preitschopf in [2].

After dimensional reduction to a 4-dimensional spacetime, structures appear that seem to be related to twistors and to Hestenes spinors.

A good general reference to twistors is the book of R. O. Wells, Jr., [16]. Rafal Ablamowicz relates twistors to Clifford algebras in [1]. Martin Cederwall discusses twistors in [3].

Pertti Lounesto describes Hestenes spinors and Clifford algebras in [6].

2 $Spin(0,8)$ and its Clifford Algebra $Cl(0,8)$

Begin with the unique triality situation of a $Spin(0,8)$ gauge group, an 8-dimensional spacetime V_8, and two mirror image 8-dimensional fermion half-spinor spaces S_8^+ and S_8^-.

They are all contained in the 256-dim positive definite Clifford algebra $Cl(0,8) = \mathbf{R}(16)$, which has the graded algebra structure:

1	8	28	56	70	56	28	8	1

0	2	2	2	2	2	2	2	7	5	5	5	5	5	5	5
4	4	2	2	2	2	2	2	5	7	5	5	5	5	5	5
4	4	4	2	2	2	2	2	5	5	7	5	5	5	5	5
4	4	4	4	2	2	2	2	5	5	5	7	5	5	5	5
4	4	4	4	4	2	2	2	5	5	5	5	7	5	5	5
4	4	4	4	4	4	2	2	5	5	5	5	5	7	5	5
4	4	4	4	4	4	4	2	5	5	5	5	5	5	7	5
4	4	4	4	4	4	4	4	5	5	5	5	5	5	5	7
1	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4
3	1	3	3	3	3	3	3	6	4	4	4	4	4	4	4
3	3	1	3	3	3	3	3	6	6	4	4	4	4	4	4
3	3	3	1	3	3	3	3	6	6	6	4	4	4	4	4
3	3	3	3	1	3	3	3	6	6	6	6	4	4	4	4
3	3	3	3	3	1	3	3	6	6	6	6	6	4	4	4
3	3	3	3	3	3	1	3	6	6	6	6	6	6	4	4
3	3	3	3	3	3	3	1	6	6	6	6	6	6	6	8

The GF and GH terms of the 8-dimensional Lagrangian

$$\int_{V_8} F_8 \wedge \star F_8 + \overline{S_8^+} \, \not\partial S_8^- + GF + GH$$

are the gauge fixing and ghost terms required for quantization.

This paper will deal with the classical terms of the 8-dimensional

Lagrangian. There, the relevant parts of $Cl(0,8)$ are:

the 1-dim 0-vector scalar part, which will be related to the Higgs scalar;

the 8-dim 1-vector vector part, which is the 8-dim spacetime V_8;

the 28-dim 2-vector bivector part, which gives the 8-dim 28 gauge bosons of the curvature term $F_8 \wedge \star F_8$, where $\star$ is the Hodge dual taking the k-vector part into the (8-k)-vector part; and

the two 8-dim half-spinor parts, which give the 8 first generation fermion particles and antiparticles.

The 8-dim half-spinors are defined with respect to the even subalgebra $Cl_e(0,8)$, which contains the even grade parts of $Cl(0,8)$:

1 28 70 28 1

0	2	2	2	2	2	2	2								
4	4	2	2	2	2	2	2								
4	4	4	2	2	2	2	2								
4	4	4	4	2	2	2	2								
4	4	4	4	4	2	2	2								
4	4	4	4	4	4	2	2								
4	4	4	4	4	4	4	2								
4	4	4	4	4	4	4	4								
								4	4	4	4	4	4	4	4
								6	4	4	4	4	4	4	4
								6	6	4	4	4	4	4	4
								6	6	6	4	4	4	4	4
								6	6	6	6	4	4	4	4
								6	6	6	6	6	4	4	4
								6	6	6	6	6	6	4	4
								6	6	6	6	6	6	6	8

The two 8×8 parts of $Cl_e(0,8)$ are the +1 and -1 eigenvalue parts of $Cl_e(0,8)$ with respect to Clifford multiplication by the 1-dimensional 8-vector volume pseudoscalar.

The + and - parts each have 8-dimensional column vector minimal left ideals. They are isomorphic, so either one (say, +) can be used as a basis for discussion in this paper. Each can be given an octonionic basis $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. They represent the 8-dimensional + and - half-spinor representations that give the first generation fermion particles and anti-particles.

The + and - parts each also have 8-dimensional row vector minimal right ideals. Each can be given an octonionic basis $\{1, i, j, k, e, ie, je, ke\}$. They represent the 8-dimensional gammas used, along with the covariant derivative, to define the Dirac operator $\not{\partial}$.

	1	i	j	k	e	ie	je	ke
1	0	2	2	2	2	2	2	2
e_1	4	4	2	2	2	2	2	2
e_2	4	4	4	2	2	2	2	2
e_3	4	4	4	4	2	2	2	2
e_4	4	4	4	4	4	2	2	2
e_5	4	4	4	4	4	4	2	2
e_6	4	4	4	4	4	4	4	2
e_7	4	4	4	4	4	4	4	4

2.1 Triality and Half-spinor - Vector Supersymmetry

Of the physically relevant parts of $Cl(0,8) = \mathbf{R}(16)$, only the 8-dimensional 1-vector vector spacetime V_8 part is not in the even subalgebra $Cl_e(0,8)$. However, by the triality automorphism among V_8, S_8^+ and S_8^-, all the physically relevant parts of $Cl(0,8) = \mathbf{R}(16)$ can be described by the even subalgebra $Cl_e(0,8)$.

0	2	2	2	2	2	2	2								
4	4	2	2	2	2	2	2								
4	4	4	2	2	2	2	2								
4	4	4	4	2	2	2	2								
4	4	4	4	4	2	2	2								
4	4	4	4	4	4	2	2								
4	4	4	4	4	4	4	2								
4	4	4	4	4	4	4	4								
1								4	4	4	4	4	4	4	4
	1							6	4	4	4	4	4	4	4
		1						6	6	4	4	4	4	4	4
			1					6	6	6	4	4	4	4	4
				1				6	6	6	6	4	4	4	4
					1			6	6	6	6	6	4	4	4
						1		6	6	6	6	6	6	4	4
							1	6	6	6	6	6	6	6	8

As is apparent from the diagram, the spacetime 1-vector V_8 can be described through triality by the same row-vector basis $\{1, i, j, k, e, ie, je, ke\}$ used to represent the 8-dimensional gammas in defining the Dirac operator $\not\partial$.

The triality representation of the vector spacetime is effectively a generalized supersymmetry between vector spacetime and the half-spinor fermions. As discussed in [12], it can be extended by the relationship between vectors and bivectors to a generalized supersymmetry between fermions and gauge bosons. It is the reason

that $Spin(0,8)$ structures are, in my opinion, the most useful structures in constructing physically realistic particle physics models.

1 28 35

	1	i	j	k	e	ie	je	ke
1	0	2	2	2	2	2	2	2
e_1	4	4	2	2	2	2	2	2
e_2	4	4	4	2	2	2	2	2
e_3	4	4	4	4	2	2	2	2
e_4	4	4	4	4	4	2	2	2
e_5	4	4	4	4	4	4	2	2
e_6	4	4	4	4	4	4	4	2
e_7	4	4	4	4	4	4	4	4

2.2 $Cl_e(0,8)$ Clifford Even Subalgebra

The 128-dim even Clifford subalgebra $Cle(0,8) = \mathbf{R}(16) + \mathbf{R}(16) = Cl(0,7)$ is made up of the sum of two 64-dimensional Clifford algebras:

$\mathbf{R}(16) + \mathbf{R}(16) = Cl(0,6) + Cl(0,6)$.

One of the two 64-dimensional $\mathbf{R}(16) = Cl(0,6)$ Clifford algebras is the + half-spinor representation of $Spin(0,8)$, and the other is the - half-spinor representation.

By the triality automorphism, it is sufficient to discuss either one of them, say, the + half-spinor of $Spin(0,8)$, which will be the 64-dimensional

$\mathbf{R}(16) = Cl(0,6)$ Clifford algebra discussed here.

As the + part of $Cl_e(0,8)$, it has the graded structure:

1 28 35

	1	i	j	k	e	ie	je	ke
1	0	2	2	2	2	2	2	2
e_1	4	4	2	2	2	2	2	2
e_2	4	4	4	2	2	2	2	2
e_3	4	4	4	4	2	2	2	2
e_4	4	4	4	4	4	2	2	2
e_5	4	4	4	4	4	4	2	2
e_6	4	4	4	4	4	4	4	2
e_7	4	4	4	4	4	4	4	4

It is important to see that, of the + part of $Cl_e(0,8)$, only
the 1-dim 0-vector part, the precursor of the Higgs scalar;
the 28-dim 2-vector part, the $Spin(0,8)$ gauge bosons,
the 8-dim half-spinor column vectors with octonionic basis
$\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$; and
the 8-dim spacetime (by triality) row vectors with
octonionic basis $\{1, i, j, k, e, ie, je, ke\}$
have physical significance in the Lagrangian.

The 35-dimensional 4-vector part is redundant.

It goes away naturally by considering $Cl(0,6)$ and its even Clifford subalgebra.

First, consider that 28-dimensional $Spin(0,8)$ is naturally represented by the upper triangular set of 2 elements on the diagram of the + part of $Cl_e(0,8)$.

Then, consider that $Spin(0,8)$ has a natural 16-dimensional $U(4)$ subgroup. If U_4 denotes an element of $U(4)$, it can be represented (see section 412 G of [5]) as an element of $Spin(0,8)$ by

$\mathbf{Re}(U_4)$	$\mathbf{Im}(U_4)$
$-\mathbf{Im}(U_4)$	$\mathbf{Re}(U_4)$

Then, the $U(4)$ subgroup of $Spin(0,8)$ can be represented on the diagram of the + part of $Cl_e(0,8)$ as

	1	i	j	k	e	ie	je	ke
1	0	U_4	U_4	U_4	U_4	U_4	U_4	U_4
e_1	4	4	U_4	U_4	2	U_4	U_4	U_4
e_2	4	4	4	U_4	2	2	U_4	U_4
e_3	4	4	4	4	2	2	2	U_4
e_4	4	4	4	4	4	2	2	2
e_5	4	4	4	4	4	4	2	2
e_6	4	4	4	4	4	4	4	2
e_7	4	4	4	4	4	4	4	4

Then, by moving some U_4 elements on or below the diagonal of the diagram of the + part of $Cl_e(0,8)$, and moving the 0-vector and some 4-vector elements above the diagonal:

	1	i	j	k	e	ie	je	ke
1	U_4	U_4	U_4	U_4	0	4	4	4
e_1	U_4	U_4	U_4	U_4	2	4	4	4
e_2	U_4	U_4	U_4	U_4	2	2	4	4
e_3	U_4	U_4	U_4	U_4	2	2	2	4
e_4	4	4	4	4	4	2	2	2
e_5	4	4	4	4	4	4	2	2
e_6	4	4	4	4	4	4	4	2
e_7	4	4	4	4	4	4	4	4

Then, by moving the 0-vector and the upper 2-vector elements on or below the diagonal of the diagram of the + part of $Cl_e(0,8)$, and moving some 4-vector elements above the diagonal:

	1	i	j	k	e	ie	je	ke
1	U_4	U_4	U_4	U_4	4	4	4	4
e_1	U_4	U_4	U_4	U_4	4	4	4	4
e_2	U_4	U_4	U_4	U_4	4	4	4	4
e_3	U_4	U_4	U_4	U_4	4	4	4	4
e_4	4	4	4	4	2	2	2	2
e_5	4	4	4	4	2	2	2	2
e_6	4	4	4	4	2	2	2	2
e_7	4	4	4	4	4	4	4	0

Then, since the 4-vector elements are not useful in the physical model, eliminate them to get the physical part of the diagram of the 64-dimensional + part of $Cl_e(0,8)$:

	1	i	j	k	e	ie	je	ke
1	U_4	U_4	U_4	U_4				
e_1	U_4	U_4	U_4	U_4				
e_2	U_4	U_4	U_4	U_4				
e_3	U_4	U_4	U_4	U_4				
e_4					2	2	2	2
e_5					2	2	2	2
e_6					2	2	2	2
e_7								0

The physical part of the + part of $Cl_e(0,8)$ is a 29-dimensional subspace of the 32-dimensional block diagonal subspace of the 64-dimensional + part of $Cl_e(0,8)$ with two 16-dimensional square blocks.

The physical part is one 16-dimensional block that is $U(4) \subset Spin(0,8)$, and a 13-dimensional subspace of the other 16-dimensional block that is the 0-vector $Spin(0,8)$ scalar plus the remaining 12 = 28 - 16 dimensions of the 2-vector adjoint bivectors of $Spin(0,8)$.

Since $U(4)/U(1) = SU(4)$, $SU_4 = Spin(0,6)$, and the + part of $Cl_e(0,8)$ $= Cl(0,6)$, it is natural at this point to look at $Cl(0,6)$.

3 $Cl(0,6) = \mathbf{R}(8)$

The + part of $Cl_e(0,8) = Cl(0,6)$, the positive definite Clifford algebra of $Spin(0,6) = SU(4) \subset U(4)$.

$Cl(0,6)$ is 64-dimensional, and has the graded structure

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

	1	i	j	k	e	ie	je	ke
1	0	2	2	2	3	5	5	5
e_1	2	2	2	2	3	3	5	5
e_2	2	2	2	2	3	3	3	5
e_3	2	2	2	2	3	3	3	3
e_4	3	3	3	3	4	4	4	4
e_5	1	3	3	3	4	4	4	4
e_6	1	1	3	3	4	4	4	4
e_7	1	1	1	3	4	4	4	6

Just as with the + part of $Cl_e(0,8)$, $Cl(0,6)$ can be given a column vector fermion particle octonionic basis $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and a row vector spacetime octonionic basis $\{1, i, j, k, e, ie, je, ke\}$.

The 2-vector bivector part of $Cl(0,6)$ is just $SU(4) \subset U(4)$, and, if the 0-vector part of $Cl(0,6)$ is taken to be $U(1) \subset U(4)$, the upper 16-dimensional block diagonal is the same $U(4)$ that is a subgroup of $Spin(0,8)$ in the + part of $Cl_e(0,8)$.

The lower 16-dimensional block diagonal of 15 4-vectors and a 6-vector can be transformed by the Hodge $\star$ to 15 2-vectors and a 0-vector.

The transformed 0-vector corresponds to the 0-vector scalar of the + part of $Cl_e(0,8)$.

Of the 15 transformed $Cl(0,6)$ 4-vectors, 12 correspond to the 12 remaining $Cl(0,8)$ 2-vectors that are outside the $U(4) \subset Spin(0,8)$, and the other 3 are not physically relevant, as they correspond to $Cl(0,8)$ 4-vectors.

As all the physically relevant parts of the + part of $Cl_e(0,8)$ are in the two 16-dimensional block diagonal parts of $Cl(0,6)$, it is now natural to look at the 32-dimensional even subalgebra $Cl_e(0,6)$ of $Cl(0,6)$.

4 $Cl_e(0,6)$ and Dimensional Reduction

$Cl_e(0,6) = Cl(0,5) = \mathbf{C}(4) = \mathbf{Re}(\mathbf{C}(4)) + \mathbf{Im}(\mathbf{C}(4))$ is 32-dimensional, and has the graded structure:

$$1 \quad 15 \quad 15 \quad 1$$

	1	i	j	k	e	ie	je	ke
1	0	2	2	2				
e_1	2	2	2	2				
e_2	2	2	2	2				
e_2	2	2	2	2				
e_4					4	4	4	4
e_5					4	4	4	4
e_6					4	4	4	4
e_7					4	4	4	6

$Cl_e(0,6)$ can be given a column vector fermion particle octonionic basis $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and a row vector spacetime octonionic basis $\{1, i, j, k, e, ie, je, ke\}$.

The 2-vector bivector part of $Cl_e(0,6)$ is just $SU(4) \subset U(4)$, and, if the 0-vector part of $Cl_e(0,6)$ is taken to be $U(1) \subset U(4)$, the upper 16-dimensional block diagonal is the same $U(4)$ that is a subgroup of $Spin(0,8)$ in the $+$ part of $Cl_e(0,8)$.

The lower 16-dimensional block diagonal of 15 4-vectors and a 6-vector can be transformed by the Hodge $\star$ to 15 2-vectors and a 0-vector.

The transformed 0-vector corresponds to the 0-vector scalar of the $+$ part of $Cl_e(0,8)$.

Of the 15 transformed $Cl_e(0,6)$ 4-vectors, 12 correspond to the 12 remaining $Cl(0,8)$ 2-vectors that are outside the $U(4) \subset Spin(0,8)$, and the other 3 are not physically relevant, as they correspond to $Cl(0,8)$ 4-vectors.

If U_4 denotes an element of $U(4) \subset Spin(0,8)$, and if 0 and 2 denote the $Spin(0,8)$ scalar 0-vector and bivector 2-vectors, a 29-dimensional subspace of 32-dimensional $Cle(0,6)$ can now be seen to correspond to the physically relevantpart of the + part of $Cl_e(0,8)$, with graded structure:

$$1 \qquad 15 \qquad 15 \qquad 1$$

	1	i	j	k	e	ie	je	ke
1	U_4	U_4	U_4	U_4				
e_1	U_4	U_4	U_4	U_4				
e_2	U_4	U_4	U_4	U_4				
e_3	U_4	U_4	U_4	U_4				
e_4					2	2	2	2
e_5					2	2	2	2
e_6					2	2	2	2
e_7								0

This 8×8 diagram of $Cl_e(0,6)$ is clearly equivalent to the following 8×4 diagram of $Cl(0,5) = Cl_e(0,6) = \mathbf{C}(4)$.

$$1 \qquad 5 \qquad 10 \qquad 10 \qquad 5 \qquad 1$$

	1	i	j	k
1	U_4	U_4	U_4	U_4
e_1	U_4	U_4	U_4	U_4
e_2	U_4	U_4	U_4	U_4
e_3	U_4	U_4	U_4	U_4
e_4	2	2	2	2
e_5	2	2	2	2
e_6	2	2	2	2
e_7				0

The dimension of row vector spacetime has been reduced from 8, with basis $\{1, i, j, k, e, ie, je, ke\}$,
to 4, with basis $\{1, i, j, k\}$.

This dimensional reduction mechanism, which can be seen as taking the octonionic basis element e of $\{1, i, j, k, e, ie, je, ke\}$ into 1, or as ignoring e as unobservable or unphysical, leaving a quaternionic 4-dimensional spacetime with basis $\{1, i, j, k\}$, has been advocated by Martin Cederwall and Corinne Manogue with respect to dimensional reduction of superstring theory from 10 dimensions to 4 dimensions, and by me [13, 14, 15] in dimensional reduction of my model from 8 dimensions to 4 dimensions.

After dimensional reduction, there are two 16-dimensional spaces in
$Cl_e(0,6) = Cl(0,5) = \mathbf{C}(4) = \mathbf{Re}(\mathbf{C}(4)) + \mathbf{Im}(\mathbf{C}(4))$:
$\mathbf{Re}(\mathbf{C}(4))$, with 16-dimensional $U(4)$; and
$\mathbf{Im}(\mathbf{C}(4))$, of which 16-dimensional space only 13 dimensions are physically relevant, the 1-dim $Spin(0,8)$ 0-vector scalar and the 12-dim remainder of $Spin(0,8)$ outside the subgroup $U(4)$.

Each of the 16-dimensional spinor spaces may have some relationship to the operator spinors of Hestenes and Keller that are described by Pertti Lounesto in [6].

4.1 $\mathbf{Re}(\mathbf{C}(4))$ - Gravity and Dirac Complexification

Consider first the $\mathbf{Re}(\mathbf{C}(4))$ part of $Cl_e(0,6) = Cl(0,5)$, with 16-dimensional $U(4)$.

As it is in the upper left diagonal block of $Cl(0,6)$, the $U(4)$ gauge group acts directly on the surviving 4-dimensional spacetime with basis $\{1, i, j, k\}$. Action on the first-generation fermion particle basis $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ is direct with respect to $\{1, e_1, e_2, e_3\}$, and defined on $\{e_4, e_5, e_6, e_7\}$ by the octonion automorphism between the two quaternionic subspaces of the octonions.

It should be noted that the $U(4)$ structure, and its related conformal structure, may be related to twistors.

The resulting structure is:

1 10 5

		1	i	j	k
1	e_4	U_4	U_4	U_4	U_4
e_1	e_5	U_4	U_4	U_4	U_4
e_2	e_6	U_4	U_4	U_4	U_4
e_3	e_7	U_4	U_4	U_4	U_4

Since $U(4)/U(1) = SU(4) = Spin(0,6)$, and $Spin(0,6)$ is the compact version of the conformal group of 4-dimensional spacetime, the 15-dimensional $SU(4) = Spin(0,6)$ can be used as a local gauge group symmetry to produce the Einstein-Hilbert action for gravity, as has been shown by Mohapatra in section 14.6 of [9].

Mohapatra's conformal group approach is similar to the approach of MacDowell and Mansouri [7] that I have used in [14, 15]. MacDowell and Mansouri used an $Sp(2) = Spin(5)$ local gauge symmetry group to get Einstein-Hilbert gravity.

Mohapatra used the same method, but used gauge fixing of the 5 conformal degrees of freedom to reduce conformal $Spin(0,6)$ to 10-dimensional $Spin(0,5)$, the bivector Lie algeba of $Cl(0,5)$.

The relation of the 5 conformal degrees of freedom to the 10 dimensions of $Spin(0,5)$ is shown in the following diagram of the even part of $Cl(0,5)$:

1 10 5

		1	i	j	k
1	e_4	0	4	4	2
e_1	e_5	4	4	2	2
e_2	e_6	4	2	2	2
e_3	e_7	2	2	2	2

The 10 $Cl(0,5)$ 2-vectors are the $Spin(0,5)$ gauge group that produces Einstein-Hilbert gravity.

The 5 $Cl(0,5)$ 4-vectors are the conformal degrees of freedom that are gauge-fixed by Mohapatra in his production of Einstein-Hilbert gravity.

If the 5 conformal degrees of freedom are not fixed, but are considered to be physical and used, the result is the quantum theory of gravity described by Narlikar and Padmanabhan in chapters 12 and 13 of [10].

The $Cl(0,5)$ **0-vector is the** $U(1) = U(4)/SU(4)$**.**

Physically, the $U(1)$ **is the Dirac complexification and it gives the physical Dirac gammas their complex structure, so that they are** $\mathbf{C}(4)$ **instead of** $\mathbf{R}(4)$**.**

There are two ways to take the even subalgebra of $Cl_e(0,6)$.
The compact, positive-definite, Euclidean way
is to use $Cl_e(0,6) = \mathbf{C}(4) = Cl(0,5)$.
Then $Cl_e(0,5) = Cl(0,4) = \mathbf{H}(2) = Cl(1,3)$.
The non-compact, anti-deSitter, Minkowski way
is to use $Cl_e(0,6) = \mathbf{C}(4) = Cl(2,3)$.
Then $Cl_e(2,3) = Cl(2,2) = \mathbf{R}(4) = Cl(3,1)$.

$\mathbf{C} \otimes \mathbf{H}(2) = \mathbf{C} \otimes \mathbf{R}(4) = \mathbf{C}(4)$,

so Dirac complexification justifies the physical use of Wick rotation between Euclidean and Minkowski spacetimes.

4.2 $\mathbf{Im}(\mathbf{C}(4))$ - $SU(3) \times SU(2) \times U(1)$ plus Higgs

Consider second the $\mathbf{Im}(\mathbf{C}(4))$ part of $Cl_e(0,6) = Cl(0,5)$, with a scalar 0-vector corresponding to the 0-vector scalar of the + part of $Cl_e(0,8)$ and

with 12 bivectors corresponding to the 12 remaining $Cl(0,8)$ 2-vectors that are outside the $U(4) \subset Spin(0,8)$.

There are 3 degrees of freedom in 16-dimensional $\mathbf{Im}(\mathbf{C}(4))$ that are not physically relevant, as they correspond to $Cl(0,8)$ 4-vectors.

As these 13 degrees of freedom are in the lower right diagonal block of $Cl(0,6)$, they acted directly on the 4-dimensional part of 8-dimensional spacetime with basis $\{e, ie, je, ke\}$ that did not survive the dimensional reduction process, called the collapsed dimensions.

Since they did not act directly on the surviving 4-dimensional spacetime with basis $\{1, i, j, k\}$, their action after dimensional reduction should be the action of a local gauge symmetry rather than the action of a spacetime symmetry.

Their action on the first-generation fermion particle basis

$\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ is

direct with respect to $\{e_4, e_5, e_6, e_7\}$, and defined on $\{1, e_1, e_2, e_3\}$ by the octonion automorphism between the two quaternionic subspaces of the octonions.

The resulting structure is:

		1	i	j	k
1	e_4	2	2	2	2
e_1	e_5	2	2	2	2
e_2	e_6	2	2	2	2
e_3	e_7				0

The 3×3 upper left block of bivectors forms a local $U(3)$ gauge group.

The 3 far right-column bivectors form a local $SU(2)$ gauge group.

The scalar 0-vector forms the Higgs scalar.

If the elements of $U(3)$, $SU(2)$, and the Higgs are denoted respectively by U_3, SU_2, and H, the following structure results:

		1	i	j	k
1	e_4	U_3	U_3	U_3	SU_2
e_1	e_5	U_3	U_3	U_3	SU_2
e_2	e_6	U_3	U_3	U_3	SU_2
e_3	e_7				H

Since $U(1) = U(3)/SU(3)$, the structure is a version of the standard model

$SU(3) \times SU(2) \times U(1)$ plus Higgs.

Details of this model, including the geometry of the Higgs mechanism and calculations of force strengths and particle masses, are given in [12, 13, 14, 15].

The $SU(3) \times SU(2) \times U(1)$ plus Higgs as described in this section, together with the conformal gravity and Dirac complexification described in the preceding section, produce a realistic and consistent model of particle physics plus gravity.

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