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Sets and $\mathbf{C^n}$; Quivers and $A-D-E$; Triality; Generalized Supersymmetry; and $D_4-D_5-E_6$

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Abstract

The relationship between mathematical Geisteswissenschaft and physics Naturwissenschaft has been discussed by Münster in hep-th/9305104 [1].

The plan of this paper is to begin with the empty set $\emptyset$; use it to form sets and quivers (sets of points plus sets of arrows between pairs of points); and then use them to make complex vector spaces and to get the $A-D-E$ Coxeter-Dynkin diagrams.

The D_n $Spin(2n)$ Lie algebras have spinor representations to describe fermions.

D_4 $Spin(8)$ triality gives automorphisms among the vector and the two half-spinor representations.

D_5 $Spin(10)$ contains both $Spin(8)$ and E_6 contains both $Spin(10)$ and the two half-spinor representations of $Spin(10)$, and therefore contains the adjoint representation of $Spin(8)$ and the complexifications of the vector and the two half-spinor representations of $Spin(8)$. E_6 is the basis for construction of a fundamental model of physics that is consistent with experiment (hep-th/9302030, hep-ph/9301210) [2].

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1 Does Naturwissenschaft Physics come from Geisteswissenschaft Mathematics?

In hep-th/9305104, Gernot Münster discusses his view of the role of mathematics in physics. Since mathematics is Geisteswissenschaft, a creation of the human mind, and physics is Naturwissenschaft, the observed phenomena of the physical world, it is remarkable that mathematics is so useful in describing physics. Münster is certainly not alone in his views, as he cites Dyson, Manin, Wigner, Courant, Hertz, and others who are similarly impressed by the deep connections between mathematics and physics.

The purpose of this paper is to try to start with *fundamental* mathematical structures,

then use *natural* operations on them to construct more elaborate structures, and

finally to observe that among these *natural fundamental* structures, there is a unique structure with a triality relation between spinor fermions and vector spacetime that induces a relation between spinor fermions and bivector bosons.

Unlike 1-1 supersymmetry, it does not require superalgebras, nor does it produce a lot of unobserved particles.

Such a structure can be used to construct a model of elementary particle physics and gravity that is consistent with experiment [2].

The key words above are *fundamental* and *natural*. It should go without saying that in this paper I will use them according to my views. I certainly recognize that others have other views and so may disagree with my choices, and that alternative choices may be as interesting and useful as mine, or perhaps more so. However, I think that the structures set out in this paper show the existence of at least one reasonable path leading from the mental constructs of fundamental mathematics to the experimental results of elementary particle physics plus gravity.

To show that the answer is YES to the question of this section "Does Naturwissenschaft Physics come from Geisteswissenschaft Mathematics?" a path from Geisteswissenschaft to Naturwissenschaft must be found.

Since the path must lead to Naturwissenschaft, and since the minimal $SU(3) \times SU(2) \times U(1)$ standard model plus gravity has been confirmed by all experiments so far, the path must be required to lead to a model containing spacetime, gauge bosons, fermion particles, and fermion antiparticles, related to each other in a way that is consistent with the Lagrangian quantum field theory of the minimal standard model, plus gravity.

Since the path should begin with Geisteswissenschaft fundamental mathematics, start with some fundamental mathematical structure and proceed according the following outline of the remainder of this paper:

2 Sets and $\mathbf{C^n}$
  2.1 Sets
  2.2 Complex n-space $\mathbf{C^n}$
3 Quivers
  3.1 Quivers of Sets and Arrows
  3.2 $\mathbf{C^n}$ Representation of Quivers
4 $A - D - E$
  4.1 Ubiquity of $A - D - E$
  4.2 A_n
  4.3 D_n
  4.4 D_4 and Triality
  4.5 E_6, E_7, and E_8
5 Triality and Generalized Supersymmetry
  5.1 A_n $SU(n+1)$ does not work.
  5.2 What about D_n $Spin(2n)$?
    5.2.1 Conventional 1-1 supersymmetry, or full spinor - bivector supersymmetry, does not work.
    5.2.2 Half-spinor - bivector supersymmetry does not work.
    5.2.3 Full spinor - vector supersymmetry almost works.
    5.2.4 Half-spinor - vector supersymmetry does work.
6 Conclusion: $D_4 - D_5 - E_6$

2 Sets and $\mathbf{C^n}$

2.1 Sets

To start with, take the empty set $\emptyset$ as fundamental.

The positive integer natural numbers **N** can be built from the empty set $\emptyset$ and the set operation {}.

N is built very rapidly in this way. After the set operation {} has been used n times, $2 \uparrow 2 \ldots \uparrow 2$ (with n 2s and n-1 uparrows) numbers have been constructed.

For example, after 5 {} operations have been used, $2 \uparrow 2 \uparrow 2 \uparrow 2 \uparrow 2 =$
$= 2 \uparrow 2 \uparrow 16 = 2^{65,536}$ numbers have been constructed.

The integers **Z**, with 0 and negative integers, then come from **N** and the inverse operation of addition, subtraction.

The rational numbers **Q** come from **Z**, excluding 0, and the inverse operation of multiplication, division.

The real numbers **R** come from continuous completion of **Q**.

The complex numbers **C** come from algebraic completion of **R**, and can also be considered as doubling the dimension of **R** with the added dimension being imaginary (the basis element being $\sqrt{-1}$).

The quaternions **H** come from doubling the dimension of **C** with the 2 added dimensions being imaginary, at the algebraic cost of losing commutativity but preserving associativity.

The octonions **O** come from doubling the dimension of **H** with the 4 added dimensions being imaginary, at the algebraic cost of losing associativity but preserving alternativity.

This completes the construction of the division algebras **R**, **C**, **H**, and **O** considered to be fundamental by Geoffrey Dixon [3].

The complex numbers **C** are the only topologically complete, algebraically complete, commutative, and associative division algebra, so **C** will be used

as the fundamental base field.

n-dimensional vector spaces $\mathbf{C^n}$ can be constructed over $\mathbf{C}$.

Starting with the empty set $\emptyset$, and then proceeding in a natural way, $\mathbf{C^n}$ has been built.

2.2 Complex n-space $\mathbf{C^n}$

Now that we have $\mathbf{C^n}$, what do we do with it? In particular, is there a naturally fundamental way to build models using vector spaces of the form $\mathbf{C^n}$?

Given a bunch of vector spaces $\mathbf{C^r}$, $\mathbf{C^s}$, ... , $\mathbf{C^t}$, the most natural thing to do is to define maps among them.

The $r \times s$ complex matrices $\mathbf{C(r,s)}$ describe $\mathbf{C}$-linear transformations from $\mathbf{C^r}$ to $\mathbf{C^s}$.

Here, I take $\mathbf{C(r,s)}$ as the most natural way to define maps among the vector spaces of the form $\mathbf{C^n}$.

I do recognize that some people would say that linear matrices are too special, and that all sorts of nonlinear maps should be introduced.

Even so, I will stick with the $r \times s$ complex matrices $\mathbf{C(r,s)}$ and proceed.

At this stage, I could build $Gl(n, \mathbf{C})$, and then easily get
$SU(3) \times SU(2) \times U(1)$ for the minimal standard model gauge group; use either $Sl(2, \mathbf{C})$ or $SU(2,2)$ for the Poincare group or the conformal Penrose twistor group for gravity; and use a Hilbert space built from $\mathbf{C^n}$ as the space of rays for quantum state vectors, acted on by complex operators built from complex Hermitian matrices.

Some might be happy with this, but I am not, because the mathematical structure does not clearly define the relations among the gauge groups, the spacetime, and the quantum states and operators, much less the identities and masses of fermion particles and antiparticles, the relative strengths of the forces, and the reason for stopping at $SU(3)$ rather than including a gauge group such as, say, $SU(17)$.

3 Quivers

3.1 Quivers of Sets and Arrows

To get more structure, I must start with something more than the empty set $\emptyset$.

First, add an arbitrary set P of points. Just a set of points by itself does not add much structure, so include a minimal amount of relations among the points in the set.

As a minimal set of relations among the points in P, use a set A of arrows between pairs of the points in P, with every arrow α in A going from a tail $\mathrm{t}(\alpha)$ in P to a head $\mathrm{h}(\alpha)$ in P.

Such a 4-tuple (P, A, t, h), where P is a set of points, A is a set of arrows between points, and, if α is an arrow, $\mathrm{t}(\alpha)$ is the point at the tail of the arrow and $\mathrm{h}(\alpha)$ is the point at the head of the arrow, has been studied in 1972 by Peter Gabriel [4], who thereby founded the theory of quivers of arrows.

A quiver (P, A, t, h) defines, and is defined by, both an (unoriented) graph Γ (by the points and their connections by arrows) and an orientation Λ (by the directions of the arrows).

Clearly, many quivers may correspond to the same (unoriented) graph Γ.

A quiver is called connected if its graph Γ is connected.

3.2 $\mathbf{C^n}$ Representation of Quivers

Gabriel represented a quiver (P, A, t, h) by representing the points in P as vector spaces and the arrows α in A as matrix maps from the vector space representing $\mathrm{t}(\alpha)$ to the vector space representing $\mathrm{h}(\alpha)$.

Here, we have the complex vector spaces $\mathbf{C^n}$ and the $r \times s$ complex matrices $\mathbf{C(r, s)}$ with which to construct such representations.

Gabriel's theorem is:

If a connected quiver has only finitely many non-isomorphic indecomposable representations, its graph is a Coxeter-Dynkin diagram of one of the Lie

algebras A_n, D_n, E_6, E_7, or E_8, and there is a 1-1 correspondence between the classes of isomorphic indecomposable representations and the positive roots of that Lie algebra. [4]

4 $A - D - E$

Using Gabriel's theorem, we now have the remarkable result that, if we start with the empty set $\emptyset$ and quivers of arrows, the only finitely representable structures we have are the $A - D - E$ Lie algebras.

4.1 Ubiquity of $A - D - E$

V. I. Arnold [5] has pointed out that the $A - D - E$ classification appears in such apparently (but not really) diverse areas as critical points of functions, Lie algebras, categories of linear spaces, caustics, wave fronts, regular polyhedra in 3-dimensional space, and Coxeter crystallographic reflection groups.

Robert Gilmore [6] has a nice description of how the E_6, E_7, and E_8 Coxeter-Dynkin diagrams correspond to the tetrahedron, octahedron, and icosahedron in 3-dimensional space. His book is a good introduction and source-book for Lie groups and related topics.

Michio Kaku [7] describes how the $A - D - E$ classification appears in superstring conformal field theory, being in 1-1 correspondence not only with the modular invariants of $SU(2)_k$, but also with the special solutions of solutions of c=1 theory for two continuous classes and the three discrete solutions. Kaku says that this is because of the correspondence between the simply laced groups and the finite subgroups of $SU(2)$.

Kaku's description indicates that, at this point along the path from Geisteswissenschaft to Naturwissenschaft, superstring conformal field theory has a claim to be a natural physics theory derived from fundamental mathematics.

I do not choose to use superstring conformal field theory as my example of a fundamental theory for two reasons:

first, there is no natural way I know of to pick a unique theory from the many possible superstring theories; and

second, all the superstring theories known to me predict the existence of a lot of particles (many of them arising from a 1-1 boson-fermion supersymmetry) that have never been experimentally observed.

Therefore, after noticing a lot of interesting mathematics along the way, I will now proceed along my chosen path.

4.2 A_n

For A_n, the $SU(n+1)$ Lie algebra, the Coxeter-Dynkin diagram has n nodes and n-1 lines:

Each node corresponds to one of the n fundamental representations of A_n $SU(n+1)$.

All irreducible representations of A_n $SU(n+1)$ can be formed from tensor products and linear combinations of the n fundamental representations (plus the trivial 1-dimensional "scalar" representation denoted by 1).

If the node on one end (by symmetry, it doesn't matter which end) of the A_n Coxeter-Dynkin diagram corresponds to the fundamental (n+1)-dimensional representation denoted by $(n+1)$, then the n-1 other fundamental representations have dimension given by antisymmetric exterior products of the (n+1)-dimensional representation space

$(n+1) \wedge (n+1) \ldots \wedge (n+1)$,

and the fundamental representations of A_n $SU(n+1)$ can be represented by the non-scalar exterior products of the complex vector space $\mathbf{C^{n+1}}$, which in turn can be represented by the non-unit terms of the (n+1) level of the Yang Hui triangle:

						1								
					1		1							
				1		**2**		1						
			1		**3**		**3**		1					
		1		**4**		**6**		**4**		1				
	1		**5**		**10**		**10**		**5**		1			
1		**6**		**15**		**20**		**15**		**6**		1		
	7		**21**		**35**		**35**		**21**		**7**		1	
8		**28**		**56**		**70**		**56**		**28**		**8**		1

In the above triangle, the bold-face entries represent the fundamental representations of the A_n $SU(n+1)$ Lie algebra through n+1 = 8.

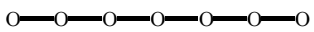
The triangular representation of the expansion of $(1+1)^n$ was invented by Yang Hui in 1261 A.D., when China was divided between the Southern Song Dynasty and the Mongol Yuan Dynasty [8]. It is widely known among Europeans as Pascal's triangle.

The adjoint representation of $U(n+1)$ is just the tensor product $(n+1) \otimes (n+1)$ of the vector and pseudovector fundamental representations of A_n, and

$$Adjoint(U(n+1)) = (n+1) \otimes (n+1) = Adjoint(SU(n+1)) \oplus 1$$

$U(n+1)$ can be represented in $\mathbf{C}((\mathbf{n+1}), (\mathbf{n+1}))$ as the $(n+1) \times (n+1)$ anti-hermitian complex matrices, with its $SU(n+1)$ subgroup being represented by those anti-hermitian complex matrices with determinant 1.

For example, A_7 $SU(8)$, with Coxeter-Dynkin diagram



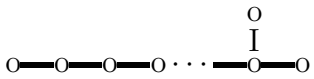
has fundamental representations of dimensions

8—28—56—70—56—28—8

The 63-dimensional adjoint representation of $SU(8)$ is determined by $8 \otimes 8 = 64 = 63 \oplus 1$

4.3 D_n

For D_n, the $Spin(2n)$ Lie algebra, the Coxeter-Dynkin diagram has n-2 nodes in a row connected with n-3 lines, with the (by usual convention, right-hand) end node connected further by 2 more lines to 2 more nodes:



Each node corresponds to one of the n fundamental representations of D_n $Spin(2n)$.

All irreducible representations of D_n $Spin(2n)$ can be formed from tensor products and linear combinations of the n fundamental representations (plus the trivial 1-dimensional "scalar" representation denoted by 1).

If the node on single node end of the D_n Coxeter-Dynkin diagram corresponds to the fundamental $2n$-dimensional vector representation of $Spin2n$ denoted by $2n$, then the n-3 other fundamental representations nearest it have dimension given by antisymmetric exterior products of the $2n$-dimensional representation space $2n \wedge 2n \ldots \wedge 2n$ and these n-2 fundamental representations of D_n $Spin(2n)$ can be represented by the non-scalar exterior products of the complex vector space $\mathbf{C^{2n}}$, which in turn can be represented by the first n-2 non-unit terms of the 2n level of the Yang Hui triangle:

														1																
												1			1															
											1			2			1													
										1			3			3			1											
									1			4			6			4			1									
								1			5			10			10			5			1						
							1			**6**			15			20			15			6			1				
						1			7			21			35			35			21			7			1		
					1			**8**			**28**			56			70			56			28			8			1	
				1			9			36			84			126			126			84			36			9			1
			1			**10**			**45**			**120**			210			252			210			120			45			10			1

In the above triangle, the bold-face entries represent the fundamental vector and multivector (i.e., non-spinor) representations of the D_n $Spin(2n)$ Lie algebra through 2n = 10.

Since 1-2=-1, D_1 $Spin(2)$ has no bold-face entry, as its 1-dimensional half-spinor representation coincides with its adjoint representation, and its 2-dimensional vector representation coincides with its reducible full spinor representation.

Since 2-2=0, D_2 $Spin(4)$ has no bold-face entry, as its 6-dimensional adjoint representation is not irreducible, $Spin(4) = SU(2) \times SU(2)$, and as its 4-dimensional vector representation coincides with its reducible full spinor representation.

Since 3-2=1, D_3 $Spin(6)$ has no bold-face entry for the 15 bivector representation, because it can be constructed for the two 4-dimensional half-spinor representations by $15 = 4 \otimes 4 \oplus 1$, and so is not irreducible with respect to tensor products and linear combinations.

The adjoint representation of $Spin(2n)$ is just the bivector exterior product $2n \wedge 2n$, of dimension n(2n-1).

The part of $Spin(2n)$ representable by the n-2 fundamental representations discussed above can be represented in $\mathbf{R}(\mathbf{2n}, \mathbf{2n})$ as the $2n \times 2n$ anti-symmetric real matrices with determinant 1.

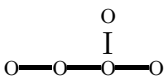
The two fundamental representations on the far end of the Coxeter-Dynkin diagram are the half-spinor representations of $Spin(2n)$, each of dimension 2^{n-1}.

As each is the mirror image of the other, they are here denoted by $+2^{n-1}$ and -2^{n-1}.

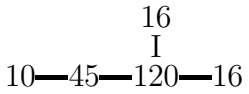
Their sum, $+2^{n-1} \oplus -2^{n-1}$, has dimension 2^n. It is the reducible full spinor representation of $Spin(2n)$, and is here denoted by 2^n.

The square $2^n \otimes 2^n$ of the full spinor representation of $Spin(2n)$ has dimension 2^{2n}, and can be represented by the entire graded Clifford algebra whose graded dimensions are given by the entire 2n level of the Yang Hui triangle.

For example, D_5 $Spin(10)$, with Coxeter-Dynkin diagram



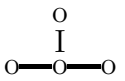
has fundamental representations of dimensions



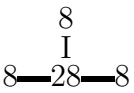
The exterior product of the two half-spinor representations, $+16 \wedge -16$, has the same dimension, 120, as the representation to which they are both connected.

4.4 D_4 and Triality

The triality relationship among the two half-spinor representations and the vector representation exits only for D_4 $Spin(8)$, with Coxeter-Dynkin diagram



D_4 $Spin(8)$ has fundamental representations of dimensions



For D_4 $Spin(8)$ the 8-dimensional vector representation is isomorphic to each of the 8-dimensional half-spinor representations.

This is the triality automorphism, a unique property of the D_4 $Spin(8)$ Lie algebra.

Also, just as in the case of D_5, the exterior product of the two half-spinor representations, $+8 \wedge -8$, has the same dimension, 28, as the representation to which they are both connected.

The Weyl group of a Lie algebra is the finite reflection group of of its root vector polytope. The Weyl group of D_4 is of order $2^{4-1} \times S_4 = 8 \times 24 = 192$.

The exceptional nature of D_4 is illustrated by its root vector polytope. Since D_4 has rank 4, the root vector polytope is 4-dimensional. It forms a 24-cell, denoted by { 3,4,3 } , that can be given quaternionic coordinates

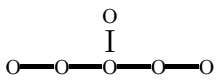
$\pm 1, \pm i, \pm j, \pm k, and (\pm 1 \pm i \pm j \pm k)/2.$

These are the unit integral quaternions, and 4-dimensional spacetime can be tiled with them, forming the D_4 lattice of integral quaternions [9].

The D_4 lattice should be useful in forming a Feynman checkerboard model in a 4-dimensional spacetime.

4.5 E_6, E_7, and E_8

For the E_6 Lie algebra, the Coxeter-Dynkin Diagram is:



Each node corresponds to one of the 6 fundamental representations of E_6. Their dimensions are:

78
I
27—351—2925—351—27

The adjoint representation of E_6 is the 78.
The two 351s are each $27 \wedge 27$.
The 2925 is $27 \wedge 27 \wedge 27$.
Note that $78 \wedge 78 = 3003 = 2925 \oplus 78$.

The Weyl group of E_6 is the finite reflection group of a 6-dimensional (not regular) polytope with 72 vertices.

It is isomorphic to the group of automorphisms of the 27 lines on the cubic surface in $\mathbf{CP^3}$. Its order is $72 \times 6! = 51840$.

The 6-dimensional lattice formed by the E_6 root vectors can be seen as the sublattice of the 8-dimensional E_8 lattice that is the set of vectors in the E_8 lattice perpendicular to any hexagonal A_2 sublattice.

The E_8 lattice is the 8-dimensional lattice of integral octonions. It can be formed from two D_8 lattices, much as the 3-dimensional structure of diamond can be formed from two D_3 lattices.

Each vertex of the E_8 lattice has 240 nearest neighbors. They do not form a regular polytope in 8 real dimensions, but they do form the regular Witting polytope in 4 complex dimensions.

The Witting complex polytope has 40 diameters which can represent the 40 root vectors of D_5 $Spin(10)$.

Each of the 40 hyperplanes orthogonal to one of the 40 diameters contains 72 vertices, which can represent the 72 root vectors of E_6.

The van Oss polygon of Witting polytope consists of the vertices in a complex plane joining a complex edge to the center. It contains 24 vertices, which can represent the 24 root vectors of D_4 $Spin(8)$.

This material on polytopes and lattices is mostly taken from Coxeter [9] and from Conway and Sloane [10].

Unlike the A_n and D_n Lie algebras, which can be represented by anti-hermitian complex matrices or by anti-symmetric real matrices, the exceptional Lie algebras E_6, E_7, and E_8 are based on non-associative octonions and are not representable in a straightforward way by associative matrices.

A rough, non-rigorous way to visualize the Lie algebras E_6, E_7, and E_8 is:

$E_6 = Spin(10) \oplus \text{Spinor}(Spin(10)) \oplus U(1)$,
where the dimensions are $78 = 45 + 32 + 1$;

$E_7 = Spin(12) \oplus \text{Spinor}(Spin(12)) \oplus SU(2)$,
where the dimensions are $133 = 66 + 64 + 3$; and

$E_8 = Spin(16) \oplus \text{Half-Spinor}(Spin(16))$,
where the dimensions are $248 = 120 + 128$.

The fundamental representations of the exceptional Lie algebra E_8 have been described by J. Frank Adams [11]. I understand that J. Frank Adams's book on exceptional Lie groups may be forthcoming in the near future.

Two good treatments of the exceptional Lie algebras are the Caltech preprint of Pierre Ramond [12] and the paper of A. Sudbery [13].

5 Triality and Generalized Supersymmetry

5.1 A_n $SU(n+1)$ does not work.

Except for the low-dimensional cases of isomorphism with D_n:
D_2 $Spin(4) = A_1 \times A_1$ $SU(2) \times SU(2)$,
and
D_3 $Spin(6) = A_3$ $SU(4)$,
the A_n $SU(n+1)$ Lie algebras have only vector or multivector fundamental representations, and do not have spinor representations.

Therefore, I do not think that the A_n $SU(n+1)$ Lie algebras are the fundamental mathematical structures on which to build a theory of physics with both bivector gauge bosons and spinor fermions.

The low-dimensional isomorphism cases are dealt with systematically under the subsection on D_n $Spin(2n)$.

5.2 What about D_n $Spin(2n)$?

The D_n $Spin(2n)$ Lie algebras all have both bivector representations that can be used for gauge bosons and two mirror image half-spinor representations that can be used for fermion particles and fermion antiparticles.

Is there a natural value of n for which the D_n $Spin(2n)$ Lie algebra can be used to build a realistic model of physics?

5.2.1 Conventional 1-1 supersymmetry, or full spinor - bivector supersymmetry, does not work.

Conventional 1-1 supersymmmetry between full spinor fermions and bivector gauge bosons is the most obvious criterion for selecting a natural value of n.

Since the spinor fermions are in the two half-spinor representations of D_n $Spin(2n)$, and each half-spinor representation has dimension 2^{n-1}, there are 2^{n-1} fermion particles and 2^{n-1} fermion antiparticles. By using the Stückelberg-Feynman interpretation of antiparticles as particles travelling

backwards in time, a supersymmetry between half-spinors and either bivector gauge bosons or vectors whose exterior product $vector \wedge vector =$ bivector gauge bosons should be just as useful as a supersymmetry between full spinors and either bivectors or vectors.

Since the gauge bosons are in the bivector representation with dimension n(2n-1), there are n(2n-1) of them.

The following table shows the numbers of half-spinors, of full spinors, and of bivector gauge bosons for the D_n $Spin(n)$ Lie algebras from n=1 to n=9:

n	$Group$	$Half-Spinors$	$Spinors$	$Gauge\ Bosons$
1	$Spin(2)$	1	2	1
2	$Spin(4)$	2	4	6
3	$Spin(6)$	4	8	15
4	$Spin(8)$	8	16	28
5	$Spin(10)$	16	32	45
6	$Spin(12)$	32	64	66
7	$Spin(14)$	64	128	91
8	$Spin(16)$	128	256	120
9	$Spin(18)$	256	512	153

If n is larger than 9, it is clear that the dimension 2^{n-1} of the half-spinor representations is much larger than either of

2n, the dimension of the vector representation, or

n(2n-1), the dimension of the bivector adjoint representation.

There is no value of n for which the number of full spinors is equal to the number of bivector gauge bosons.

Therefore, conventional 1-1 fermion-boson supersymmmetry, or
full spinor - bivector supersymmetry, does not give a natural value of n.

5.2.2 Half-spinor - bivector supersymmetry does not work.

Using the Stückelberg-Feynman interpretation of antiparticles as particles travelling backwards in time, try to identify half-spinors 1-1 with bivector gauge bosons.

From the table, that works only for D_1 $Spin(2) = U(1)$ acting on a 2-dimensional vector spacetime.

Therefore, Geisteswissenschaft indicates that a $Spin(2) = U(1)$ gauge field theory over a 2-dimensional spacetime is natural, and indeed a uniquely nice model can be built with D_1:

the 2-dimensional Dirac equation with its natural Feynman checkerboard.

However, a $U(1)$ gauge field theory over a 2-dimensional spacetime is obviously inadequate to describe Naturwissenschaft.

The conformal field theory used in superstrings can be said to be based on D_1, but it does not produce a single unique theory that is consistent with Naturwissenschaft experiments.

Therefore conventional supersymmetry of fermions with bivector gauge bosons has failed to satisfy the demands of Naturwissenschaft experiments.

5.2.3 Full spinor - vector supersymmetry almost works.

Next, consider a generalized supersymmetry between fermions and vectors, with the relationship to bivector gauge bosons determined by bivector = $vector \wedge vector$.

A 1-1 symmetry between full spinors and vectors, according to the table, works only for D_2 $Spin(4) = SU(2) \times SU(2)$ acting on a 4-dimensional vector spacetime.

Since $Spin(4)$ is the Lie algebra of the Lorenz group, and Naturwissenschaft spacetime is 4-dimensional, the 1-1 full spinor - vector symmetry does give special relativity and the 4-dimensional Dirac gamma matrices (full spinors) for fermions.

Also, since $Spin(4) = SU(2) \times SU(2)$ is reducible to two copies of $SU(2)$, the 1-1 full spinor - vector symmetry indicates that SU(2) gauge field theories over a 4-dimensional base manifold spacetime should be interesting and useful, as they are.

However, the D_2 $Spin(4)$ models are not elaborate enough to explain by themselves the Naturwissenschaft phenomena of $SU(3)$ color gauge forces and the 8 types of fermion particles: electron; e-neutrino; red, blue, and green up quarks; and red, blue, and green down quarks.

Even though I don't think that D_2 models are big enough to explain Naturwissenschaft, there are at least two lines of work that have a chance at doing so successfully.

The earlier effort is that of Heisenberg and his coworkers, including Durr and Saller, [14] to construct a unified field theory based on $SU(2)$, an irreducible component of D_2 $Spin(4) = SU(2) \times SU(2)$.

The Heisenberg approach deals with the $SU(3)$ color force and quarks by noting that the gluons and quarks are not asymptotic states, but protons and pions are. Therefore, protons and pions should be represented by nonlinear states, such as solitons, with the color force being represented as a higher-order force generated by a fundamental urfield derived from $SU(2)$.

Whether or not the Heisenberg approach is successful in all details, I think that it is correct in considering protons and pions as soliton-type states, thus explaining the success of phenomenological models such as bag models and non-relativistic potential models using constituent quark masses.

The later effort is that of Hestenes, Keller, et. al. [15]. They notice that full spinors are defined as 2^n-dimensional minimal ideals in 2^{2n}-dimensional Clifford algebras, so that full spinors can be defined as either left-ideals or right-ideals.

In terms of $2^n \times 2^n$ matrices, the full spinors can be chosen to be either $2^n \times 1$ column vectors or 1×2^n row vectors.

To use the whole Clifford algebra, both the left-ideal column spinors and the right-ideal row spinors should be used.

They then use the left-ideal column spinors to define the spinor transformations of the vector spacetime, and the right-ideal row spinors to define the fermion spinor particles and antiparticles.

I think that their general approach is correct, but that the 4-dimensional full spinors of D_2 $Spin(4)$ are not big enough for the 4-dimensional right-ideal row spinors to account for all 8 types of fermion particles: electron; e-neutrino; red, blue, and green up quarks; and red, blue, and green down quarks.

5.2.4 Half-spinor - vector supersymmetry does work.

Using both the Stückelberg-Feynman interpretation of antiparticles as particles travelling backwards in time, and a generalized supersymmetry between fermions and vectors, with the relationship to bivector gauge bosons determined by bivector = $vector \wedge vector$, try to identify half-spinors 1-1 with the vector representation.

From the table, that works only for in the unique case with the property of triality: D_4 $Spin(8)$ acting on an 8-dimensional vector spacetime.

To make a Naturwissenschaft physics model from the 4 fundamental representations of D_4 $Spin(8)$, it is natural to construct a Lagrangian gauge field model with

the 8-dimensional vector representation as spacetime,

the 8-dimensional +half-spinor representation as the fermion particles,

the 8-dimensional half-spinor representation as the fermion anti-particles, and

the 28-dimensional bivector representation as the gauge bosons.

The 8-dimensional vector spacetime can be reduced to a 4-dimensional spacetime.

Prior to dimensional reduction, the generalized supersymmetry relationship between the 28 gauge bosons and the 8-dimensional spaces of fermion particles and antiparticles *might* give an ultraviolet finite model.

In this connection, it is useful to note that, in an 8-dimensional spacetime, the dimension of each of the 28 gauge bosons in the Lagrangian is 1, and

the dimension of each of the 8 fermion particles is 7/2, so that

the total dimension of the gauge bosons is equal to the total dimension of the fermion particles, since $28 \times 1 = 8 \times 7/2$.

After dimensional reduction of spacetime to 4 dimensions,

the fermions get a 3-generation structure and

the gauge bosons are decomposed by Weyl group symmetries to produce

U(1) electromagnetism,

SU(3) color force,

SU(2) weak force plus a minimal Higgs field for weak symmetry breaking, and

a Spin(5) = Sp(2) gauge field that can produce gravity by the MacDowell-Mansouri mechanism.

6 Conclusion: $D_4 - D_5 - E_6$

The program of half-spinor - vector generalized supersymmetry has already been worked out in some detail [2] (including hep-th/9302030 and hep-ph/9301210).

In particular, the structure of complex homogeneous domains of hermitian symmetric spaces related to the various gauge groups is used to calculate the relative strengths of the forces, giving results such as $\alpha_E = 1/137.03608$ (see hep-th/9302030).

To get this structure, it is more natural to use the complex and conformal structure
$E_6 = Spin(10) \oplus \text{Spinor}(Spin(10)) \oplus U(1) =$
$= Spin(8) \oplus (\mathbf{C} \otimes \text{Vector}(Spin(8))) \oplus U(1) \oplus (\mathbf{C} \otimes \text{Spinor}(Spin(8))) \oplus U(1)$

than it is to use the real structure
$F_4 = Spin(9) \oplus \text{Spinor}(Spin(9)) =$
$= Spin(8) \oplus \text{Vector}(Spin(8)) \oplus \text{Spinor}(Spin(8))$.

E_6 unites all 4 fundamental representations of D_4 $Spin(8)$ using complex conformal structure, as:

E_6 is the conformal group whose Lorenz group is D_5 $Spin(10)$, with the quotient space being the rank 2 projective geometry of the spinor space of D_5 $Spin(10)$, which spinor space in turn can be represented as the complexification of the spinor space of D_4 $Spin(8)$; and

D_5 $Spin(10)$ is the conformal group whose Lorenz group is D_4 $Spin(8)$, with the quotient space being the Lie sphere geometry of the complexification of the vector space of D_4 $Spin(8)$.

F_4 unites all 4 fundamental representations of D_4 $Spin(8)$ using real structure, as:

the quotient space F_4 / $Spin(9)$ is the Cayley projective plane, which represents the full spinor space of $Spin(9)$, which in turn can represent the full spinor space of D_4 $Spin(8)$, and

the quotient space $Spin(9)$ / $Spin(8)$ is the 8-sphere S^8, which represents the vector space of D_4 $Spin(8)$.

Roughly speaking, F_4 represents the 4 fundamental representations of D_4 $Spin(8)$ in terms of the real geometry of an ordinary sphere and a rank-1 projective space, while E_6 represents them in terms of the complex geometry of a Lie sphere and a rank-2 projective space.

The E_6 representation is the natural and correct one, but I did not realize that until recently.

My earlier work used the F_4 representation because its structure was simpler, and then I didn't know any better. My F_4 model gave the same results as the E_6 model because I added on the complex structure "ad hoc" rather than realizing that it was the natural consequence of using fundamental sets, quivers, and $A - D - E$.

Therefore, the quantitative results of earlier papers [2], including but not limited to $\alpha_E = 1/137.03608$ and a t-quark mass of 130 GeV,
come from a D_4 - D_5 - E_6 model of Naturwissenschaft physics that is the natural consequence of Geisteswissenschaft structures:

the empty set $\emptyset$ and the operation $\{\}$;

quivers of sets and arrows;

the $A - D - E$ classification;

triality; and

half-spinor - vector generalized supersymmetry.

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