

Hermitian Jordan Triple Systems, the Standard Model plus Gravity, and $\alpha_E = 1/137.03608$

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Abstract

A physical interpretation is given for some Hermitian Jordan triple systems (HJTS) that were recently discussed by Günaydin [4].

The quadratic Jordan algebras derived from HJTS provide a formulation of quantum mechanics (Günaydin[3]) that is a natural framework within which exceptional structures are identified with physically realistic structures of a quantum field theory that includes both the standard model and MacDowell-Mansouri [14] gravity.

The structures allow the calculation of the relative strengths of the four forces, including $\alpha_E = 1/137.03608$.

1 Introduction

1.1 Overview

Since the purpose of this paper is to give a physical quantum field theoretical interpretation of the algebraic structures (Jordan algebras, Lie algebras, symmetric spaces, bounded homogeneous domains, etc.) the field over which the structures are defined is the field \mathbf{C} of the complex numbers.

In particular, the initial fundamental Lie algebras will be complexifications $G^{\mathbf{C}}$ of real Lie algebras G .

Real structures, such as spacetime, will emerge in the physical interpretation by such mechanisms as by taking the Silov boundary of a bounded complex homogeneous domain.

The remainder of this first section of this paper, *Introduction*, will be devoted to an overview of the algebraic and geometric structures to be used. Also, the work of Günaydin[3] comparing the quantum mechanical formalisms of quadratic and bilinear Jordan algebras with the conventional matrix-operator on Hilbert space formalism is used to justify the quadratic Jordan algebra approach taken in this paper.

The second section, *Physical Interpretation*, will deal with the physical interpretation of some specific structures that are taken to be fundamental.

In the third section, *Calculation of Observable Quantities*, explicit calculations will be made of physically measurable quantities, such as the electromagnetic fine structure constant $\alpha_E = 1/137.03608$.

Since the structures of this paper are defined over the field of complex numbers, the difference between Minkowski signature (d-1, 1) and Euclidean signature (d, 0) is not immediately apparent, because Wick rotation of complex spaces can transform back and forth between Minkowski and Euclidean signature.

However, after working with the structures (using such processes as taking Silov boundaries of bounded complex homogeneous domains) to get some physical structures such as spacetime, structures defined over the real numbers will emerge.

For example, the Lie algebra of a physical gauge group will be the real Lie algebra G underlying its complexification $G^{\mathbb{C}}$.

In such cases, signature of structures can be defined. In this paper, calculations of physically observable quantities use the ratios of volumes of geometric spaces. Therefore, the spaces should be represented in forms in which their volumes are finite. Instead of non-compact forms arising from vector spaces with Minkowski signature $(d-1, 1)$, compact forms arising from vector spaces with Euclidean signature $(d, 0)$ are used.

It is assumed in this paper that any physics requiring spaces with Minkowski signature can be obtained from the spaces with Euclidean signature by Wick rotation of complex spaces (which are taken to be fundamental).

1.2 Jordan Triple Systems

A Jordan triple system is defined by Satake [1] as a finite-dimensional vector space V of dimension d (usually complex in this paper) with a trilinear triple product map

$$\{\} : V \times V \times V \rightarrow V$$

such that for all $a, b, x, y, z \in V$:

$$\{x, y, z\} = \{z, y, x\}$$

and

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$$

1.3 Bilinear and Quadratic Jordan Algebras

When there is a unit element e , the triple product can be used to define two other different products:

- a bilinear product $x \circ y = \{x, y, e\} = \{x, e, y\} = \{e, x, y\}$; and
- a quadratic product $U_x y = \{x, y, x\}$.

In terms of the ordinary matrix product xy for vectors x and y in the vector space V , the products are (up to factors of 2 or $\frac{1}{2}$) given by McCrimmon [2] as:

Triple product:

$$\{x, y, z\} = xyz + zyx$$

Bilinear product:

$$x \circ y = \frac{1}{2}(xy + yx)$$

Quadratic product:

$$U_x y = xyx$$

As McCrimmon [2] states, the bilinear and quadratic Jordan algebras are categorically equivalent when there exists a scalar $\frac{1}{2}$.

1.4 Quadratic Jordan Algebra Quantum Mechanics

Günaydin [3] discusses the use of either the bilinear Jordan product or the quadratic Jordan product to formulate quantum mechanics in an alternative way to the usual Hilbert space formulation.

The following chart taken from Günaydin [3] compares various quantities in the three formulations:

<i>Hilbert Space</i>	<i>Bilinear Jordan</i>	<i>Quadratic Jordan</i>
$ \alpha\rangle$	$ \alpha\rangle\langle\alpha = P_\alpha$	P_α
$H \alpha\rangle$	$H \circ P_\alpha$	$\Pi_H P_\alpha = \{HP_\alpha H\}$
$\langle\alpha H \beta\rangle$?	?
$\langle\alpha H \alpha\rangle$	$Tr H \circ P_\alpha$	$Tr \Pi_{P_\alpha} H = Tr \{P_\alpha H P_\alpha\}$
$ \langle\alpha H \beta\rangle ^2$	$Tr P_\alpha \circ \{HP_\beta H\}$ $= Tr \{HP_\alpha H\} \circ P_\beta$	$Tr \Pi_{P_\alpha} \Pi_H P_\beta$ $= Tr \Pi_{P_\beta} \Pi_H P_\alpha$
$[H_1, H_2]$?	$4(\Pi_{H_1} \circ \Pi_{H_2} - \Pi_{H_1 \circ H_2})$

Neither the bilinear Jordan formulation nor the quadratic Jordan formulation has a natural equivalent to the Hilbert space formulation of the transition matrix element $\langle\alpha|H|\beta\rangle$.

Since $\langle \alpha | H | \beta \rangle$ is not measurable, the two Jordan formulations are closer to physical reality than the Hilbert space formulation.

In the bilinear Jordan formulation, there is no natural direct analog of the commutator of two observables $[H_1, H_2]$. According to Günaydin [3], that problem is usually managed by requiring the associator of the two observables with all elements of the bilinear Jordan algebra to vanish.

As Günaydin [3] states, there is an analog of the commutator in the quadratic Jordan formulation:

$$4(\Pi_{H_1} \circ \Pi_{H_2} - \Pi_{H_1 \circ H_2})$$

Therefore, in the quadratic Jordan formulation two observables are compatible when $\Pi_{H_1} \circ \Pi_{H_2} = \Pi_{H_1 \circ H_2}$.

In this paper, the quadratic Jordan formulation of quantum mechanics is used for the physical interpretation of Jordan triple systems.

1.5 Jordan Pairs: Conformal and Lorentz Algebras

McCrimmon [2] says that Jordan triple systems with vector space V are in 1-1 correspondence with Jordan pairs (V_+, V_-) with involution such that $V_+ = V_- = V$.

If V is d -dimensional then the Jordan pairs (V_+, V_-) with involution can be used to construct the conformal Lie algebra of the vector space V , following the approaches discussed by Günaydin [4] and by McCrimmon [2]. (Here, anticipating the need for compact Euclidean spaces in calculations, the d -dimensional vector space V is represented by S^d , the d -dimensional sphere.):

the translations of $V = S^d$ are $V_+ = S_+^d$;

the special conformal transformations of $V = S^d$ are $V_- = S_-^d$;

the Lorentz transformations and dilatations of $V = S^d$ are the Lie bracket $[V_+, V_-] = [S_+^d, S_-^d]$; and

the full conformal Lie algebra is the 3-graded Lie algebra

$$\begin{aligned} V_- \oplus [V_+, V_-] \oplus V_+ &= \\ &= S_-^d \oplus [S_+^d, S_-^d] \oplus S_+^d \end{aligned}$$

A conventional example (using real vector spaces and uncomplexified real Lie algebras) is the conformal Lie algebra $Spin(6)$ where the vector space V is 4-dimensional with Euclidean signature $(4, 0)$:

$$\begin{aligned} Spin(6) &= \\ &= S_-^4 \oplus (Spin(4) \otimes U(1)) \oplus S_+^4 \end{aligned}$$

Günaydin [4] describes the conformal Lie algebras and Lorentz Lie algebras of simple Jordan algebras.

A similar description (substituting some compact forms for non-compact ones) is as follows, denoting by $J_n^{\mathbf{C}}(A)$ and by $\Gamma(d)$ the Jordan algebra of $n \times n$ Hermitian matrices over the division algebra A and the Jordan algebra of Dirac gamma matrices in d complex dimensions, and denoting the four division algebras (real numbers, complex numbers, quaternions, and octonions) by the symbols \mathbf{R} , \mathbf{C} , \mathbf{H} , and \mathbf{O} .

This chart shows:

the Jordan (grade -1) algebra;

the Lorentz (grade 0) Lie algebra plus the dilatation algebra $U^{\mathbf{C}}(1)$;

the other member of the Jordan pair, the Jordan (grade +1) algebra; and

the conformal Lie algebra that is the sum of all three grades.

<i>Jordan</i> (-1)	<i>Lorentz</i> \oplus <i>Dilatation</i> (0)	<i>Jordan</i> (+1)	<i>Conformal</i>
$J_n^{\mathbf{C}}(\mathbf{R})$	$SU^{\mathbf{C}}(n) \oplus U^{\mathbf{C}}(1)$	$J_n^{\mathbf{C}}(\mathbf{R})$	$Sp^{\mathbf{C}}(2n)$
$J_n^{\mathbf{C}}(\mathbf{C})$	$SU^{\mathbf{C}}(n) \times SU^{\mathbf{C}}(n) \oplus U^{\mathbf{C}}(1)$	$J_n^{\mathbf{C}}(\mathbf{C})$	$SU^{\mathbf{C}}(2n)$
$J_n^{\mathbf{C}}(\mathbf{H})$	$SU^{\mathbf{C}}(2n) \oplus U^{\mathbf{C}}(1)$	$J_n^{\mathbf{C}}(\mathbf{H})$	$SO^{\mathbf{C}}(4n)$
$J_3^{\mathbf{C}}(\mathbf{O})$	$E_6^{\mathbf{C}} \oplus U^{\mathbf{C}}(1)$	$J_3^{\mathbf{C}}(\mathbf{O})$	$E_7^{\mathbf{C}}$
$\Gamma^{\mathbf{C}}(n)$	$Spin^{\mathbf{C}}(n+1) \oplus U^{\mathbf{C}}(1)$	$\Gamma(n)$	$Spin^{\mathbf{C}}(n+3)$

1.6 Hermitian Jordan Triple Systems (HJTS)

As McCrimmon [2] states, a Jordan triple system is an Hermitian Jordan triple system (HJTS) if the triple product $\{x, y, z\}$ is \mathbf{C} - *linear* in x and z but \mathbf{C} - *antilinear* in y , and the bilinear form $\langle x, y \rangle = \text{Tr } V_{x,y}$, where $V_{x,y} : z \rightarrow \{x, y, z\}$, is a positive definite Hermitian scalar product.

There exist four infinite families of HJTS and two exceptional ones. They are given by Günaydin [4] as:

Type I _{p,q} generated by $p \times q$ complex matrices $M_{p,q}(\mathbf{C})$ with the triple product

$$(abc) = ab^\dagger c + cb^\dagger a \quad (1.1)$$

where \dagger represents the usual hermitian conjugation.

Type II _{n} generated by complex anti-symmetric $n \times n$ matrices $A_n(\mathbf{C})$ with the triple product

$$(abc) = ab^\dagger c + cb^\dagger a \quad (1.2)$$

where \dagger represents the usual hermitian conjugation.

Type III _{n} generated by complex $n \times n$ symmetric matrices $S_n(\mathbf{C})$ with the triple product

$$(abc) = ab^\dagger c + cb^\dagger a \quad (1.3)$$

where \dagger represents the usual hermitian conjugation.

Type IV _{n} generated by Dirac gamma matrices $\Gamma^{\mathbf{C}}(n)$ in n dimensions with complex coefficients and the Jordan triple product

$$(abc) = a \cdot (\bar{b} \cdot c) + c \cdot (\bar{b} \cdot a) - (a \cdot c) \cdot \bar{b} \quad (1.4)$$

where the bar $\bar{}$ denotes complex conjugation.

Type V generated by 1×2 complexified octonionic matrices $M_{1,2}^{\mathbf{C}}(\mathbf{O})$ with the triple product

$$(abc) = \{(a\bar{b}^\dagger)c + (\bar{b}a^\dagger)c - \bar{b}(a^\dagger c)\} + \{a \leftrightarrow c\} \quad (1.5)$$

where \dagger denotes octonion conjugation times transposition and the bar $\bar{}$ denotes complex conjugation.

Type VI generated by the exceptional Jordan algebra of 3×3 hermitian complexified octonionic matrices $J_3^{\mathbf{C}}(\mathbf{O})$ with the triple product

$$(abc) = a \cdot (\bar{b} \cdot c) + c \cdot (\bar{b} \cdot a) - (a \cdot c) \cdot \bar{b} \quad (1.6)$$

where the bar $\bar{}$ denotes complex conjugation.

The simple HJTS, their Jordan algebras, their Lorentz Lie algebras, and their Conformal Lie algebras are as follows (see Günaydin [4] and McCrimmon [2]):

<i>HJTS</i>	<i>Jordan</i>	<i>Lorentz</i>	<i>Conformal</i>
$S_n(\mathbf{C})$	$J_n^{\mathbf{C}}(R)$	$SU^{\mathbf{C}}(n)$	$Sp^{\mathbf{C}}(2n)$
$M_{p,q}(\mathbf{C})$	(If $p = q = n$) : $J_n^{\mathbf{C}}(C)$	$SU^{\mathbf{C}}(p) \times SU^{\mathbf{C}}(q)$	$SU^{\mathbf{C}}(p + q)$
$A_n(\mathbf{C})$	$J_n^{\mathbf{C}}(H)$	$SU^{\mathbf{C}}(n)$	$SO^{\mathbf{C}}(2n)$
$J_3^{\mathbf{C}}(\mathbf{O})$	$J_3^{\mathbf{C}}(\mathbf{O})$	$E_6^{\mathbf{C}}$	$E_7^{\mathbf{C}}$
$\Gamma^{\mathbf{C}}(n)$	$\Gamma^{\mathbf{C}}(n)$	$Spin^{\mathbf{C}}(n + 1)$	$Spin^{\mathbf{C}}(n + 3)$
$M_{1,2}^{\mathbf{C}}(\mathbf{O})$	$M_{1,2}^{\mathbf{C}}(\mathbf{O})$	$Spin^{\mathbf{C}}(10)$	$E_6^{\mathbf{C}}$

1.7 HJTS are 1-1 with Bounded Homogeneous Circled Domains

McCrimmon states in Theorem 6.7 of [2], there is a 1-1 correspondence between HJTS and bounded homogeneous circled domains.

1.8 HJTS are 1-1 with Hermitian Symmetric Spaces

There is a 1-1 correspondence among HJTS, bounded homogeneous circled domains and Hermitian symmetric spaces, provided that the Hermitian symmetric spaces of compact type are considered to be in the same equivalence class as their noncompact dual Hermitian symmetric spaces. [1, 2, 5]

For the purpose of physical interpretation in this paper, an HJTS is considered to be identified by the 1-1 correspondences just mentioned with the corresponding bounded homogeneous circled domain and with the corresponding compact Hermitian symmetric space.

1.9 Which HJTS should be taken as Fundamental?

If all the above HJTS can be used as the basis for a quantum theory, which one should be taken as the fundamental structure for interpretation as a physically realistic theory?

Dixon [6], noting that the only good division algebras are \mathbf{R} , \mathbf{C} , \mathbf{H} , and \mathbf{O} , and that nature ought to use all the good things that are available, has advocated using the 64(real) dimensional space $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$ as the fundamental structure.

Since I like that approach, I note that the closest thing to $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$ that is an HJTS is the HJTS of *Type VI*, $J_3^{\mathbf{C}}(\mathbf{O})$, the 3×3 Hermitian matrices of octonions over the complex number field.

$J_3^{\mathbf{C}}(\mathbf{O})$ is 54(real) dimensional, and can be represented as the complexification of the following 27(real) dimensional matrix, where \mathbf{O}_+ , \mathbf{O}_- , \mathbf{O}_v are octonion, a, b, c are real, and \dagger denotes octonion conjugation:

$$\begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & c \end{pmatrix}$$

For details about the 27(real) dimensional Jordan algebra $J_3(\mathbf{O})$ see, for example, Adams [7].

If $J_3^{\mathbf{C}}(\mathbf{O})$ were to be equal to $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$, the matrix structure would have to be equivalent to quaternionic structure. Of course, it is not, but it contains the imaginary part of the quaternions in the sense that the imaginary quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are contained in the following 3(real) dimensional rotation matrices:

$$\begin{pmatrix} a & \mathbf{O}_+ & 0 \\ \mathbf{O}_+^\dagger & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & \mathbf{O}_- \\ 0 & \mathbf{O}_-^\dagger & c \end{pmatrix} \begin{pmatrix} a & 0 & \mathbf{O}_v \\ 0 & b & 0 \\ \mathbf{O}_v^\dagger & 0 & c \end{pmatrix}$$

Since the *Type VI* HJTS $J_3^{\mathbf{C}}(\mathbf{O})$ carries the most of $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$ that is consistent with Jordan algebra structure, the physical interpretation in this paper starts with it, and with its corresponding *Type VI* Hermitian symmetric space of compact type:

the set of $(\mathbf{C} \otimes \mathbf{O})P^2$'s in $(\mathbf{H} \otimes \mathbf{O})P^2$;
that is, $\frac{E_7}{E_6 \times U(1)}$ with 54(real) dimensions. [5, 8]

Note that the *Type VI* Hermitian symmetric space

$$\frac{E_7}{E_6 \times U(1)} = J_3^{\mathbf{C}}(\mathbf{O})$$

has 54 real dimensions,

whereas the Jordan pair for the *Type VI* HJTS is

$$\frac{E_7^{\mathbf{C}}}{E_6^{\mathbf{C}} \times U^{\mathbf{C}}(1)} = J_3^{\mathbf{C}}(\mathbf{O}) \oplus J_3^{\mathbf{C}}(\mathbf{O})$$

which has 54 complex dimensions, or 108 real dimensions.

Details of the physical interpretation are discussed in the next section.

2 Physical Interpretation

2.1 HJTS of *Type VI*: $J_3^{\mathbf{C}}(\mathbf{O})$

The full conformal Lie algebra corresponding to the HJTS of *Type VI* is the 3-graded Lie algebra

$$\begin{aligned} E_7^{\mathbf{C}} &= \\ &= J_3^{\mathbf{C}}(\mathbf{O}) \oplus (E_6^{\mathbf{C}} \times U^{\mathbf{C}}(1)) \oplus J_3^{\mathbf{C}}(\mathbf{O}) \end{aligned}$$

The translations and the special conformal transformations are each represented by the space $J_3^{\mathbf{C}}(\mathbf{O})$ of the HJTS of *Type VI*.

2.1.1 Space of Operators = $J_3^{\mathbf{C}}(\mathbf{O})$

The space of operators of the quadratic Jordan algebra quantum theory is identified with the 54(real) dimensional space $J_3^{\mathbf{C}}(\mathbf{O})$.

2.1.2 Provisional Space of States = $J_3^{\mathbf{C}}(\mathbf{O})$

The provisional candidate for the quantum state space is $J_3^{\mathbf{C}}(\mathbf{O})$ itself.

This choice takes advantage of one distinction between the quadratic Jordan algebra approach and the Hilbert space matrix operator approach:

in the quadratic Jordan algebra approach, states are projection operators P_α and the action of the operator on the state is the quadratic Jordan product $\Pi_H P_\alpha = \{HP_\alpha H\}$;

in the Hilbert space matrix operator approach, the states $|\alpha\rangle$ is a 3×1 (octonionic) vector space and the action of the operator 3×3 matrices on the state is the matrix product $H|\alpha\rangle$.

The provisional state space is therefore $J_3^{\mathbf{C}}(\mathbf{O})$, which can be represented as the following 54(real) dimensional matrix, where $\mathbf{O}_+^{\mathbf{C}}, \mathbf{O}_-^{\mathbf{C}}, \mathbf{O}_v^{\mathbf{C}}$ are complexified octonions, $a^{\mathbf{C}}, b^{\mathbf{C}}, c^{\mathbf{C}}$ are complexified real numbers, and \dagger denotes octonion conjugation.

(Note that here the complexification was done after the 27(real) dimensional $J_3(\mathbf{O})$ was formed by using the \dagger of octonion conjugation for Hermitianness. This is different from forming $J_3((\mathbf{C} \otimes \mathbf{O}))$ by using both the \dagger of octonion conjugation and the $\bar{}$ of complex conjugation for Hermitianness.

Also note that the quadratic Jordan product in $J_3^{\mathbf{C}}(\mathbf{O})$, and the triple product from which it is derived, do use the $\bar{}$ of complex conjugation.)

$$\begin{pmatrix} a^{\mathbf{C}} & \mathbf{O}_+^{\mathbf{C}} & \mathbf{O}_v^{\mathbf{C}} \\ \mathbf{O}_+^{\mathbf{C}\dagger} & b^{\mathbf{C}} & \mathbf{O}_-^{\mathbf{C}} \\ \mathbf{O}_v^{\mathbf{C}\dagger} & \mathbf{O}_-^{\mathbf{C}\dagger} & c^{\mathbf{C}} \end{pmatrix}$$

The provisional physical interpretation of the elements of the 3×3 matrix $J_3^{\mathbf{C}}(\mathbf{O})$ is:

$\mathbf{O}_+^{\mathbf{C}}$ represents the (first generation) fermion particles. If a basis (with complex scalars) for $\mathbf{O}_+^{\mathbf{C}}$ is $\{1, i, j, k, e, ie, je, ke\}$, then the representation is

$\mathbf{O}_+^{\mathbf{C}}$ <i>basis element</i>	<i>Fermion Particle</i>
1	<i>e – neutrino</i>
<i>i</i>	<i>red up quark</i>
<i>j</i>	<i>green up quark</i>
<i>k</i>	<i>blue up quark</i>
<i>e</i>	<i>electron</i>
<i>ie</i>	<i>red up quark</i>
<i>je</i>	<i>green up quark</i>
<i>ke</i>	<i>blue up quark</i>

$\mathbf{O}_-^{\mathbf{C}}$ represents the (first generation) fermion antiparticles in the same physically realistic way as the fermion particles were represented;

$\mathbf{O}_v^{\mathbf{C}}$ represents an 8-dimensional spacetime, which can be reduced to 4 dimensions as described in [10, 11]; and

$a^{\mathbf{C}}, b^{\mathbf{C}}, c^{\mathbf{C}}$ are not given a physical interpretation.

2.1.3 Difficulties with Provisional Space of States = $J_3^{\mathbf{C}}(\mathbf{O})$

The provisional space of states $J_3^{\mathbf{C}}(\mathbf{O})$ is not physically realistic. It has the following four difficulties.

1. The quadratic Jordan algebra action on $J_3^{\mathbf{C}}(\mathbf{O})$ mixes the elements $\mathbf{O}_+^{\mathbf{C}}$ and $\mathbf{O}_-^{\mathbf{C}}$ (fermion particles and antiparticles) with $\mathbf{O}_v^{\mathbf{C}}$ (spacetime).

2. The elements $a^{\mathbf{C}}, b^{\mathbf{C}}, c^{\mathbf{C}}$ have no physical interpretation.

3. The space $J_3^{\mathbf{C}}(\mathbf{O})$ is complex, not real.

The 8(complex) dimensional spacetime $\mathbf{O}_v^{\mathbf{C}}$ must be converted into an 8(real) dimensional spacetime prior to its dimensional reduction (as described in [10, 11]) to a physically realistic 4(real) dimensions.

4. The space $J_3^{\mathbf{C}}(\mathbf{O})$ is not compact.

To do calculations, compact spaces are needed so that finite volumes can be calculated.

The requirement of compact spaces is also consistent with the use of compact projective ray spaces as state spaces in the formalism of matrix operators on Hilbert space.

To get around the first two of these difficulties, it would be nice to have as space state a subspace of $J_3^{\mathbf{C}}(\mathbf{O})$ like:

$$\left(\begin{array}{ccc} 0 & \mathbf{O}_+^{\mathbf{C}} & 0 \\ \mathbf{O}_+^{\mathbf{C}\dagger} & 0 & \mathbf{O}_-^{\mathbf{C}} \\ 0 & \mathbf{O}_-^{\mathbf{C}\dagger} & 0 \end{array} \right) \oplus \left(\begin{array}{ccc} 0 & 0 & \mathbf{O}_v^{\mathbf{C}} \\ 0 & 0 & 0 \\ \mathbf{O}_v^{\mathbf{C}\dagger} & 0 & 0 \end{array} \right)$$

To find such a subspace, consider the HJTS of *Type V*: $M_{1,2}^{\mathbf{C}}(\mathbf{O})$.

2.2 HJTS of *Type V*: $M_{1,2}^{\mathbf{C}}(\mathbf{O})$

The full conformal Lie algebra corresponding to the HJTS of *Type V* is the 3-graded Lie algebra

$$\begin{aligned} E_6^{\mathbf{C}} &= \\ &= M_{1,2}^{\mathbf{C}}(\mathbf{O}) \oplus (Spin^{\mathbf{C}}(10) \otimes U^{\mathbf{C}}(1)) \oplus M_{1,2}^{\mathbf{C}}(\mathbf{O}) \end{aligned}$$

The translations and the special conformal transformations are each represented by the space $M_{1,2}^{\mathbf{C}}(\mathbf{O})$ of the HJTS of *Type V*.

Since $M_{1,2}^{\mathbf{C}}(\mathbf{O})$ can be represented as

$$\left(\begin{array}{ccc} 0 & \mathbf{O}_+^{\mathbf{C}} & 0 \\ \mathbf{O}_+^{\mathbf{C}\dagger} & 0 & \mathbf{O}_-^{\mathbf{C}} \\ 0 & \mathbf{O}_-^{\mathbf{C}\dagger} & 0 \end{array} \right)$$

It seems to be a good candidate for the part of the space of states representing the fermion particles and antiparticles.

In particular, the fermion particles and antiparticles can be mixed, as they should; spacetime is not mixed, as it should not be; and there are no $a^{\mathbf{C}}, b^{\mathbf{C}}, c^{\mathbf{C}}$ that have no physical interpretation.

However, $M_{1,2}^{\mathbf{C}}(\mathbf{O})$ is complex and is not compact, and so suffers from the third and fourth difficulties.

To get a real and compact space of states, consider the Silov boundary of the bounded homogeneous circled domain corresponding to the Hermitian symmetric space of *Type V*.

2.2.1 Fermion Space of States = $(\mathbf{R}P^1 \times S^7) \times (\mathbf{R}P^1 \times S^7)$

The *Type V* Hermitian symmetric space of compact type is Rosenfeld's elliptic projective plane, the irreducible Kähler manifold

$$(\mathbf{C} \otimes \mathbf{O})P^2 = \frac{E_6}{Spin(10) \times U(1)}$$

with 32 real dimensions. [5, 8]

The bounded homogeneous circled domain corresponding to the *Type V* HJTS is an irreducible bounded complex domain of 32 real dimensions.

Its 16(real) dimensional Silov boundary, denoted here by $S_+ \times S_-$, is two copies of $\mathbf{R}P^1 \times S^7$ [2, 5, 9], or

$$\begin{aligned} S_+ \times S_- &= \\ &= (\mathbf{R}P^1 \times S^7) \times (\mathbf{R}P^1 \times S^7) \end{aligned}$$

$S_+ = S_- = \mathbf{R}P^1 \times S^7$ is the compact 8(real) dimensional space of states that represents the 8 (first generation) fermion particles (S_+) and antiparticles (S_-).

If $\{1, i, j, k, e, ie, je, ke\}$ is a basis (with real scalars) for $\mathbf{R}P^1 \times S^7$, where the real axis $\{1\}$ spans $\mathbf{R}P^1$ and $\{i, j, k, e, ie, je, ke\}$ spans S^7 , then

Since $\mathbf{R}P^1$ and S^7 are both parallelizable, so is the whole space $\mathbf{R}P^1 \times S^7$; and

Since $\mathbf{R}P^1$ is double covered by S^1 , so that if $\mathbf{R}P^1$ is parameterized by $U(1)$ then S^1 is parameterized by $U(1) \times U(1)$:

Then, the representation of the (first generation) particles by $\mathbf{R}P^1$ and S^7 as state space is

$\mathbf{R}P^1 \times S^7$ <i>basis element</i>	<i>Fermion Particle</i>	<i>Phase</i>
1	<i>e – neutrino</i>	$U(1)$
<i>i</i>	<i>red up quark</i>	$U(1) \times U(1)$
<i>j</i>	<i>green up quark</i>	$U(1) \times U(1)$
<i>k</i>	<i>blue up quark</i>	$U(1) \times U(1)$
<i>e</i>	<i>electron</i>	$U(1) \times U(1)$
<i>ie</i>	<i>red up quark</i>	$U(1) \times U(1)$
<i>je</i>	<i>green up quark</i>	$U(1) \times U(1)$
<i>ke</i>	<i>blue up quark</i>	$U(1) \times U(1)$

The neutrino only has one $U(1)$ phase, and so should be a Weyl fermion existing in only one helicity state.

The electron and quarks have $U(1) \times U(1)$ phases, and so should be Dirac fermions existing in both left- and right-handed helicity states.

The (first generation) fermion antiparticles are represented similarly, except as mirror images of the particles.

The second and third generations of fermion particles and antiparticles are formed by the spacetime dimensional reduction mechanism as discussed in [10, 11].

What about the spacetime part of the space of states?

The fermion part of the space of states was found within the translation (or, equivalently, the special conformal) part of the conformal Lie algebra of the HJTS of *Type V*.

That leaves the Lorentz part (and the $U(1)$) of the conformal Lie algebra of the HJTS of *Type V*.

The Lorentz part of the conformal Lie algebra of the HJTS of *Type V* turns out to be the conformal Lie algebra of an HJTS of *Type IV₇*.

To find the spacetime part of the state space, look at the translation (or, equivalently, the special conformal) part of the conformal Lie algebra of the HJTS of *Type IV₇*.

The Lorentz part of the conformal Lie algebra of the HJTS of *Type IV₇* will then be available to construct the Lie algebra of the physical gauge group, which can be shown to act in a physically on the fermion part of the space of states just defined.

2.3 HJTS of *Type IV₇*: $J(\Gamma^{\mathbf{C}}(7))$

The exceptional HJTS of *Type IV₇* is the Jordan algebra $J(\Gamma^{\mathbf{C}}(7))$ generated by the Dirac gamma matrices $\Gamma^{\mathbf{C}}(7)$ in 7 dimensions with complex coefficients and the Jordan triple product

$$(abc) = a \cdot (\bar{b} \cdot c) + c \cdot (\bar{b} \cdot a) - (a \cdot c) \cdot \bar{b} \quad (2.7)$$

where the bar $\bar{}$ denotes complex conjugation. [4]

Since Clifford algebras are universal associative algebras, and since the Jordan algebra $J(\Gamma^{\mathbf{C}}(7))$ is associative, $J(\Gamma^{\mathbf{C}}(7))$ is contained in the even subalgebra of the Clifford algebra $Clf(\mathbf{C}^8)$, whose spin group is $Spin^{\mathbf{C}}(8)$ [12, 13].

The full conformal Lie algebra corresponding to the HJTS of *Type IV₇* is the 3-graded Lie algebra

$$\begin{aligned} Spin^{\mathbf{C}}(10) &= \\ &= J(\Gamma^{\mathbf{C}}(7)) \oplus (Spin^{\mathbf{C}}(8) \otimes U^{\mathbf{C}}(1)) \oplus J(\Gamma^{\mathbf{C}}(7)) \end{aligned}$$

.

2.3.1 Spacetime Space of States = $\mathbf{R}P^1 \times S^7$

The spacetime part of the space of states should come from the translations of the conformal Lie algebra of the HJTS of *Type IV₇*.

The translations and the special conformal transformations are each represented by the space $J(\Gamma^{\mathbf{C}}(7))$ of the HJTS of *Type IV₇*.

The *Type IV₇* Hermitian symmetric space of compact type is the irreducible Kähler manifold

$$\frac{Spin(10)}{Spin(8) \times U(1)}$$

with 16 real dimensions [5, 8].

The bounded homogeneous circled domain corresponding to the *Type IV₇* HJTS is an irreducible bounded complex domain of 16 real dimensions.

Its 8(real) dimensional Silov boundary, denoted here by V_8 , is $\mathbf{R}P^1 \times S^7$ [2, 5, 9].

The 8(real) dimensional spaces $V_8 = S_+ = S_- = \mathbf{R}P^1 \times S^7$ are all isomorphic, as is required by the triality property of $Spin(8)$ and $J_3^{\mathbf{C}}(\mathbf{O})$.

2.3.2 Gauge Group = $Spin(8)$

The physical gauge group should be defined by the Lorentz Lie algebra of the conformal Lie algebra of the HJTS of *Type IV₇*, which is $Spin^{\mathbf{C}}(8)$.

The complex structure of the Lie algebra $Spin^{\mathbf{C}}(8)$ gives the infinitesimal generators of the $Spin(8)$ gauge group a complex $U(1)$ phase, as is physically realistic for gauge boson propagators.

2.3.3 Effect of Dimensional Reduction on $Spin(8)$ Gauge Group

As is discussed in [10], the effect of the dimensional reduction of spacetime from 8(real) dimensions to 4(real) dimensions on the gauge group $Spin(8)$ is to reorganize the 28 generators of $Spin(8)$ according to Weyl group symmetry into the 10 generators of $Spin(5)$, the 8 generators of $SU(3)$, the 6 generators of $Spin(4)$, and the 4 generators of $U(1)^4$.

The $Spin(5)$, which is isomorphic to $Sp(2)$, which is in some of the literature denoted by $Sp(4)$, then gives gravity by the MacDowell-Mansouri mechanism [14];

The $SU(3)$ gives the color force;

The $Spin(4) = SU(2) \times SU(2)$ gives the $SU(2)$ weak force and, by integrating out the other $SU(2)$ over the 4(real) dimensions that are eliminated by reduction of spacetime from 8(real) dimensions to 4(real) dimensions, gives a Higgs scalar field for the Higgs mechanism; and

The $U(1)^4$ gives the 4 covariant components of the electromagnetic photon.

Details of the above are in [10].

The details include such things as description of the nonstandard relationship between the weak force and electromagnetism, calculation of the Weinberg angle, etc.

3 Calculation of Observable Quantities

The method for calculating force strengths is based on the relationships of the four gauge groups $Spin(5)$, $SU(3)$, $Spin(4)$, and $U(1)^4$ to the fundamental compact fermion and spacetime state space manifolds.

Details are in [10]. A sketch is given here.

3.1 Basis for Calculation

The calculated strength of a force is taken to be proportional to the product of four factors:

$$\left(\frac{1}{\mu^2}\right) (Vol(M)) \left(\frac{Vol(Q)}{Vol(D)^{\frac{1}{m}}}\right)$$

where

μ is a symmetry breaking mass scale factor that is the Planck mass for gravity,
the weak symmetry breaking scale for the weak force, and
1 for the color force and electromagnetism, in which symmetry breaking does not occur.

$Vol(M)$ is the volume of the irreducible m (real)-dimensional symmetric space on which the gauge group acts naturally as a component of 4(real) dimensional spacetime $M^{\frac{4}{m}}$.

The M manifolds for the gauge groups of the four forces are:

<i>Gauge Group</i>	<i>Symmetric Space</i>	<i>m</i>	<i>M</i>
$Spin(5)$	$\frac{Spin(5)}{Spin(4)}$	4	S^4
$SU(3)$	$\frac{SU(3)}{SU(2) \times U(1)}$	4	CP^2
$SU(2)$	$\frac{SU(2)}{U(1)}$	2	$S^2 \times S^2$
$U(1)$	$U(1)$	1	$S^1 \times S^1 \times S^1 \times S^1$

$Vol(Q)$ is the volume of that part of the full compact fermion state space manifold $\mathbf{RP}^1 \times S^7$ on which a gauge group acts naturally through its charged (color or electromagnetic charge) gauge bosons.

For the forces with charged gauge bosons,

$Spin(5)$ gravity (prior to action of the Macdowell-Mansouri mechanism),
 $SU(3)$ color force, and
 $SU(2)$ weak force,

Q is the Silov boundary of

the bounded complex homogeneous domain D that corresponds to

the Hermitian symmetric space on which the gauge group acts naturally as a local isotropy (gauge) group.

For $U(1)$ electromagnetism, whose photon carries no charge, the factors $Vol(Q)$ and $Vol(D)$ do not apply and are set equal to 1.

The volumes $Vol(M)$, $Vol(Q)$, and $Vol(D)$ are calculated with M, Q, D normalized to unit radius.

The factor $\frac{1}{Vol(D)^{\frac{1}{m}}}$ is a normalization factor to be used if the dimension of Q is different from the dimension m , in order to normalize the radius of

Q to be consistent with the unit radius of M .

The Q and D manifolds for the gauge groups of the four forces are:

<i>Gauge Group</i>	<i>Hermitian Symmetric Space</i>	<i>Type of D</i>	m	Q
$Spin(5)$	$\frac{Spin(7)}{Spin(5) \times U(1)}$	IV_5	4	$\mathbf{R}P^1 \times S^4$
$SU(3)$	$\frac{SU(4)}{SU(3) \times U(1)}$	B^6 (ball)	4	S^5
$SU(2)$	$\frac{SU(3)}{SU(2) \times U(1)}$	IV_3	2	$\mathbf{R}P^1 \times S^2$
$U(1)$	—	—	1	—

3.2 Results of Calculations

The relative strengths of the four forces can be calculated from the formula

$$\left(\frac{1}{\mu^2}\right) (Vol(M)) \left(\frac{Vol(Q)}{Vol(D)^{\frac{1}{m}}}\right)$$

The $\frac{1}{\mu^2}$ factor is only applicable to the weak force and gravity.
For the weak force,

$$\frac{1}{\mu^2} = \frac{1}{m_{W^+}^2 + m_{W^-}^2 + m_Z^2}$$

For gravity,

$$\frac{1}{\mu^2} = \frac{1}{m_{Planck}^2}$$

The geometric force strengths, that is, everything but the symmetry breaking mass scale factors, are

$$(Vol(M)) \left(\frac{Vol(Q)}{Vol(D)^{\frac{1}{n}}} \right)$$

They are normalized by dividing them by the largest one, the one for gravity.

The geometric volumes needed for the calculations, mostly taken from Hua [9], are

<i>Force</i>	<i>M</i>	<i>Vol(M)</i>	<i>Q</i>	<i>Vol(Q)</i>	<i>D</i>	<i>Vol(D)</i>
<i>gravity</i>	S^4	$8\pi^2/3$	$\mathbf{RP}^1 \times S^4$	$8\pi^3/3$	IV_5	$\pi^5/2^45!$
<i>color</i>	\mathbf{CP}^2	$8\pi^2/3$	S^5	$4\pi^3$	B^6 (ball)	$\pi^3/6$
<i>weak</i>	$S^2 \times S^2$	$2 \times 4\pi$	$\mathbf{RP}^1 \times S^2$	$4\pi^2$	IV_3	$\pi^3/24$
<i>e – mag</i>	T^4	$4 \times 2\pi$	–	–	–	–

Using these numbers, the results of the calculations are the relative force strengths at the characteristic energy level of the generalized Bohr radius of each force:

<i>Gauge Group</i>	<i>Force</i>	<i>Characteristic Energy</i>	<i>Geometric Force Strength</i>	<i>Total Force Strength</i>
$Spin(5)$	<i>gravity</i>	$\approx 10^{19} GeV$	1	$G_G m_{proton}^2 \approx 5 \times 10^{-39}$
$SU(3)$	<i>color</i>	$\approx 245 MeV$	0.6286	0.6286
$SU(2)$	<i>weak</i>	$\approx 100 GeV$	0.2535	$G_W m_{proton}^2 \approx 1.02 \times 10^{-5}$
$U(1)$	<i>e – mag</i>	$\approx 4 KeV$	1/137.03608	1/137.03608

The force strengths are given at the characteristic energy levels of their forces, because the force strengths run with changing energy levels.

The effect is particularly pronounced with the color force.

In [10] the color force strength was calculated at various energies according to renormalization group equations, with the following results:

<i>Energy Level</i>	<i>Color Force Strength</i>
245MeV	0.6286
5.3GeV	0.166
34GeV	0.121
91GeV	0.106

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Historical Note: As far as I know the first attempt to calculate α_E by using ratios of volumes of structures related to conformal groups and bounded homogeneous complex domains was by Armand Wyler (A. Wyler, *Arch. Ration. Mech. Anal.* **31**, 35 (1968), A. Wyler, *Acad. Sci. Paris, Comptes Rendus* **269A**, 743 (1969); A. Wyler, *Acad. Sci. Paris, Comptes Rendus* **272A**, 186 (1971): see also *Physics Today* (August 1971) 17; *Physics Today* (November 1971) 9; and R. Gilmore, *Phys. Rev. Lett.* **28**, 462 (1972)).