FUNCTIONAL ANALYSIS AND OPTIMIZATION

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Preface

The contents of this volume, better than any comments by the Editor, will show how great and fruitful is the contribution that a skillful use of modern methods of functional analysis can bring to the study and solution of challenging problems posed by recent developments in science and technology.

Linear mathematics is of little or no avail in a realistic study of stability conditions, optimal performance of systems with feedback, organizational principles in assemblies of communicating elements, etc. Thus, a whole new branch of mathematics has come into existence for the study of this vast class of phenomena, in short, optimization.

As is always the case with all beginnings, many results were at first derived with too narrow a scope, or with methods of only heuristic value. The intervention of functional analysis permits the recognition of the underlying unity of many seemingly different problems, opens new, powerful ways of attack, as well as enlarging the horizon of the pure mathematicians, who always find stimulation and incentive in an effective interaction with colleagues from other fields of natural science.

The purpose of this 7th International School of Ravello (June 1965), which the generous support of NATO made possible, was twofold: to hear eminent specialists speak on the general state of the art and of their own work; to bring together active researchers of the interested areas and have them pose problems to the mathematicians and find with them and among themselves a common language and understanding.

The scientific organization and direction of the School are due to Professor J. L. Lions, to whom belongs the credit for the gratifying success of this initiative, with our warmest personal thanks. Thanks are also due to Academic Press for their ever-efficient and understanding cooperation.

Naples, Italy E. R. CAIANIELLO
December, 1966

The First Nondistributive Algebra, with Relations to Optimization and Control Theory

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I. Introduction

Within the limitations of time and of space that circumstances have made unavoidable, only a brief outline of this ramified subject can be made here. The reader interested in further details is referred to a forthcoming monograph, "Theory of Nondistributive Multiplication and Algebraic Structures" (Muses).

Matrices are isomorphic with hypercomplex numbers; and projective geometry, affine geometry, the theory of vector spaces (hence that of function spaces), and linear algebra are all isomorphic, forming one of the most significant and profound convergences of meaning in all mathematics.

This convergence, in one or more of its aspects, is treated, for example, in the following works: H. Weyl, "Mathematische Analyse des Raumproblems" (1923); B. Segré, "Lezioni di geometria moderna" (Vol. I, 1948); G. Birkhoff and S. MacLane, "A Survey of Modern Algebra" (1948); C. C. MacDuffee, "Vectors and Matrices" (1943); O. Schreier and E. Sperner, "Introduction to Modern Algebra and Matrix Theory" (first English edition, 1951); and H. Schwerdtfeger, "Introduction to Linear Algebra and Matrix Theory" (1951). A. A. Albert's "Modern Higher Algebra" (1937) and "Structures of Algebras" (1939) are highly recommended as well.

Full awareness of the fact and significance of the above-mentioned convergence has not yet been realized, however; nor has the fact that hypercomplex numbers and their related rings and fields are the most economical way to represent it. Group theory is related to the same fundamental idea, its algebra stemming from that of matrices, and hence from linear algebras, i.e., the algebra of hypercomplex numbers.

In his modern classic, "Algebraic Theory of Numbers" (1940), Hermann Weyl observed on p. 222: "... enormous progress has been made in the theory of class fields ... but in spite of all efforts I have the feeling that the theory has not yet assumed its final form."

The reason for Weyl's penetrating appraisal is found, for instance, in the predicament of G. Voronoi, who in his theory of cubic number fields was forced to speak of ideals that were "existent" and of those that "have no real existence", by which it turns out that he means representable (or not) in terms of roots of complex numbers. The entire theory of ideals, and hence of class fields, suffers from the same defect: the limitations of complex numbers.

It is as if we said that 3 was an absolute prime because it has no explicit factors, complex or real. But it is less well known that $3=(1+i+j)\cdot(1-i-j)$ than that the number 5 factors into $(2+i)$ and $(2-i)$. For 7 we need complete quaternion factors, i.e., $7=(2+i+j+k)\cdot(2-i-j-k)$. No more than quaternion factors are required for any real integral prime, and thus H-algebra is sufficient to "dissolve" all such real primes. However, if we stop there, we need to hypostatize "ideals" (together with their modern and rather artificial extensions, "ideles" and "adeles") to eliminate apparent paradoxes arising from the investigation of complex and hypercomplex primes. But we need not stop there; and since $\sqrt{-1}$ is the point at which all number systems remain open, we are indeed forced to go on to nonassociative and then to nondistributive algebra.

Just as real algebra is not closed, but open at the operation "square root of minus unity," so are Gaussian and quaternion algebra similarly open, for $\sqrt{-1}$ is *multivalued*, i.e., j, k, etc., also satisfy it, leading us from real (R) to Gaussian (G) to quaternion or Hamiltonian (H) to Cayley (C) algebra, and beyond.

Thus hypercomplex algebra contains complex or Gaussian algebra, and the latter contains real algebra; or $R \subset G \subset H \subset C \subset N$. The last term in this sequence is new, and previous attempts to extend Cayley algebra have failed because of not realizing the theorems that: (1) any extension of R must have nonamalgamative addition; (2) any extension of G must have noncommutative multiplication; (3) any extension of H must have nonassociative multiplication; and (4) any extension of C must have nondistributive multiplication. Thus all linear algebras with less than 5 and more than 2 i-elements are contained in H; all with more than 3 and less than 8, in C; all with more than 7 and less than 16, in N. It is in this sense that R, G, H, C, and N are the only complete linear algebras with less than 17 elements in all, for all linear algebras contain the real element $i_{\pm 0} = 1^{\pm 1} = 1$, as well as the self-orthogonal elements $i_{\pm\infty} = 0^{\pm 1} = 0$, ∞i_n.

In general, $i_{\pm n} = i_n^{\pm 1} = \pm i_n$; and $i_n^2 = -1$ $(0 > n < \infty)$, where $i \equiv i_1$ $j \equiv i_2$, $k \equiv i_3$. The above nonarbitrary notation and its properties are indispensable in handling hypercomplex numbers efficiently and usefully. A table of all the complete linear algebras with finite factor multiplication follows, zero being regarded not as a finite, but as an infinitesimal number.

A complete algebra has 2^n distinct elements, none of which represents merely the same nonreal element extended along a different real dimension. Clifford algebras, since they apply i_1 to successively higher real dimensions in order to form their 2^n elements, are not complete algebras. Thus the only complete algebra requiring a representation space of four dimensions is quaternion algebra (H): the only complete algebra similarly requiring eight dimensions is octave algebra (C); and the only complete algebra similarly requiring sixteen dimensions is N-algebra. A characteristic of complete algebras is successively to require a more precise formulation of what is meant by addition or multiplication, and that each embeds in itself all the complete algebras below it, thus preserving the self-consistency of mathematics. The rules of arithmetic really do not "break down"; they merely become more sensitive, taking more distinctions into account in higher algebras. A *complete algebra* is also one whose elements form a kind of multiplication loop, regenerating each other, except for zero formation when the number of elements exceeds 8. Finite factor (ff) multiplication is that which does not entail the equation $a \cdot 0 = b$, where a, $b \neq 0$, ∞, although it may involve $a \cdot b = 0$, $a \neq b$, as in N-algebra.

Thus non-ff multiplication entrains what may be termed *zero revival*, whereas the nondistributive multiplication of N-algebra involves mutual annihilation or zero creation. To perform ff multiplication, the precise kind of zero product must be specified, the rules for such specification (deducible from the wave-operator interactions) being necessary in the process of zero revival, i.e., the operation $a \cdot 0 = b$; a, $b \neq 0$, ∞. All non-ff multiplication is nondistributive. The converse is not true, however, since there are four forms (R, G, H, and C) of ff multiplication that are distributive.

TABLE I

FIVE COMPLETE LINEAR ALGEBRAS, INVOLVING THREE KINDS OF HYPERCOMPLEX NUMBERS

Algebra			No. of elements	No. of units	Characterization: Addition	Characterization: Multiplication	
Real	(*di*)	R	1, 0	$T_1 = 2$	*am*	*co, as, di*	
Gaussian	(*di*)	G	1, 1	$(2/3)^a T_2 = 4$	$-am$	*co, as, di*	($-am$)
Hamiltonian	(*di*)	H	1, 3	$T_4 = 24$	$-am$	$-co$, *as, di*	($-am$)
Cayleyan	(*di*)	C	1, 7	$T_8 = 240$	$-am$	$-co$, $-as$, *di*	($-am$)
Nondistributive		N	1, 15	$T_{16} = 4320^b$	$-am$	$-co$, $-as$, $-di$	($-am$)

[a] This fraction is not *ad hoc* but involves the Eisenstein integers based on the cube roots of minus and plus unity, and involving the imaginary angles $\pm(2/3)\pi i$.

[b] This is the probable figure based on the densest packing yet achieved [see Leech (*14*)] for D_{16}. However, this figure does not stand on the same footing as the prior four, since the polytope of centers is no longer convex, and there is no longer distributive multiplication, which in geometric terms means that in the D_{16} lattice pack, every sphere of T_{15} no longer touches the D_{16} portions of T_{16}. This phenomenon appears from D_9 onward, since D_8 is the last dimension to have a convex *polytope of centers*, which is the D_n figure with its vertices at the centers of the hyperspheres in the T_n shell, which centers in turn may be regarded as an infinitesimal portion of a ${}_KD_{n-1}$ space where $K = 1$ if $2 \leq n \leq 8$, and $K < 0$ if $\infty > n > 9$. The T_n refer to maximal-contact tangent sphere shells in D_n.

Thus R, G, H, C, and N may be considered to have Euclidean dimensionalities of 1, 2, 4, 8, and 16, respectively, T_n being the maximal number of equal tangent hyperspheres that can be fit about another such sphere of the same radius in D_n, a parabolic space of n dimensions. In the same terminology ${}_{\pm K}D_n$ is an elliptic (+) or hyperbolic (−), i.e., a convex or concave space of curvature K and n dimensions. Since ${}_{\pm K}D_{n-1}$ is always embedded in ${}_0D_n \equiv D_n$, the parabolic, flat, or Euclidean spaces are of prime importance, and hence the linear algebras.

There is a close connection between the groups of tangent hyperspheres mentioned in the last paragraph and error-correcting computer codes, which in itself suggests the importance of the present theme for optimization and control theory. (See also pages 200ff.)

We now have the complete algebras R, G, H, C, N, each of whose multiplications is embedded toward the right; i.e., N-algebra contains R, G, H, and C. N is the first algebra with nondistributive ($-di$) as well as nonassociative ($-as$) and noncommutative ($-co$) multiplication; $(-)am$ denotes (non)amalgamative addition, only R being *am*, *co*, *as*, and *di*, just as N only in this fundamental table of the real multiplication linear algebras is characterized completely negatively as $-am$, $-co$, $-as$, and $-di$.

Before commenting further on these characterizations and on the meaning of the T_n and their relation to optimization and control, we shall append the other tables necessary to complete the multiplication table for N-algebra.

II. The A_N Multiplication Table (Unlisted triplets are $-as$)

(*a*) *The Linear Triplet* (*as*, *co*): $n \cdot n = -0$

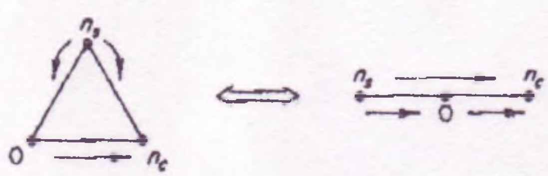


FIG. 1. Thus $n_s \cdot 0 = n_c$; $0 \cdot n_c = n_s$; $n_s \cdot n_c = -0$; i.e., $i_n \cdot i_n = -i_0 = -1$.

This linear triplet, as our diagram shows, involves by the theory of its multiplication the distinction of a separating center or source (n_s) and a combining center or sink (n_c). Thus in $i \cdot i = -1$, the two (i)'s are not identical. This subtle distinction does not affect ordinary hypercomplex algebra.

Subscript rules:

$$\left.\begin{array}{l} (1)\ -i_n = i_{-n} = i_n^{-1}, \qquad 0 < |n| < \infty. \\ (2)\ \pm i_0 = \pm i_{-0} = \pm 1^{\pm 1} = \pm 1. \\ (3)\ \pm i_\infty = \pm 0 i_n = 0; \qquad \pm i_{-\infty} = \pm (0 i_n)^{-1} = \pm \infty i_n \end{array}\right\} i_{\pm n} = i_n^{\pm 1}$$

Thus $i_{-\infty}$ is doubly indeterminate, and here $0 \leq n < \infty$.

(b) *The* H-*type Triplets* (*as*, −*co*), *Each with* 6 *Variants*

[1]1.	$3\cdot 2 = -1$	19.	$3\cdot 13 = 14$
[1a]2.	$4\cdot 1 = -5$	20.	$12\cdot 8 = 4$
[1a]3.	$7\cdot 6 = -1$	21.	$13\cdot 9 = -4$
4.	$9\cdot 1 = 8$	22.	$10\cdot 14 = -4$
5.	$10\cdot 1 = -11$	23.	$15\cdot 4 = 11$
6.	$13\cdot 12 = -1$	24.	$8\cdot 13 = 5$
7.	$14\cdot 1 = -15$	25.	$12\cdot 9 = -5$
[1a]8.	$6\cdot 4 = 2$	26.	$10\cdot 5 = -15$
[1a]9.	$7\cdot 2 = 5$	27.	$5\cdot 11 = 14$
10.	$8\cdot 10 = -2$	28.	$14\cdot 8 = 6$
11.	$11\cdot 2 = 9$	29.	$15\cdot 6 = 9$
12.	$12\cdot 14 = -2$	30.	$10\cdot 6 = 12$
13.	$13\cdot 15 = 2$	31.	$6\cdot 11 = 13$
[1a]14.	$4\cdot 7 = 3$	32.	$15\cdot 7 = 8$
[1a]15.	$5\cdot 6 = 3$	33.	$7\cdot 9 = 14$
16.	$11\cdot 3 = 8$	34.	$7\cdot 10 = 13$
17.	$10\cdot 3 = 9$	35.	$7\cdot 11 = 12$
18.	$15\cdot 12 = -3$		

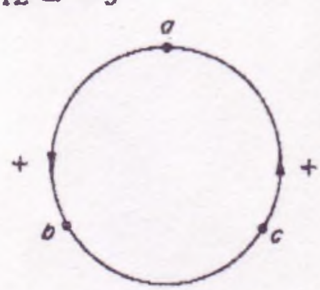


FIG. 2. The H-type or cyclic triplet, $ab = c$, etc., $ba = -c$, etc. Counterclockwise is taken as the positive sense.

Each of the above equations represents a set of six (see Fig. 2), obtainable by cyclic permuting. Since each such triplet is isomorphic to an H-algebra, which in turn is isomorphic to a D_3 sphere with a D_4 axis of rotation, such triplets may be also termed H-spheres.

[1] Pertains to H as well. The equations are indicial throughout.

[1a] Pertain to C. All others pertain to N only.

(c) *The Partially Cyclic Triplets* $(as, -co)$

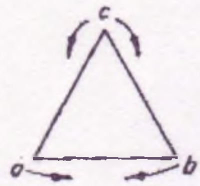


FIG. 3.

1.	2	6	$\underline{8}$	9.	$\underline{3}$	10	13	17.	6	13	$\underline{3}$	25.	10	14	$\underline{8}$
2.	2	$\underline{7}$	9	10.	3	11	$\underline{14}$	18.	6	14	$\underline{4}$	26.	10	$\underline{15}$	9
3.	2	10	$\underline{12}$	11.	$\underline{4}$	6	10	19.	6	$\underline{15}$	5	27.	11	$\underline{13}$	8
4.	2	$\underline{11}$	13	12.	$\underline{4}$	14	2	20.	$\underline{6}$	7	13	28.	$\underline{11}$	14	9
5.	3	$\underline{5}$	8	13.	5	6	$\underline{11}$	21.	7	11	$\underline{2}$	29.	$\underline{12}$	14	10
6.	$\underline{3}$	6	9	14.	$\underline{5}$	7	12	22.	7	$\underline{13}$	4	30.	13	14	$\underline{11}$
7.	3	7	$\underline{10}$	15.	5	14	$\underline{3}$	23.	$\underline{7}$	14	5				
8.	3	$\underline{9}$	12	16.	6	$\underline{12}$	2	24.	$\underline{9}$	11	4				

For example $(2\cdot 6)8 = 2(6\cdot 8)$, each triplet specifying four such equations. Underscores indicate apex members. See Fig. 3.

Triplets excluded by the scheme are $(-as)$, *e.g.*, $(6\cdot 8)2 = -6(8\cdot 2)$.

III. The (-*di*) Mutual Annihilators

Just as commutators are definable as

$$co \equiv (a, b) = ab - ba$$

and associators as

$$as \equiv (a, b, c) = (ab)c - a(bc),$$

we may also define a "distributor" as

$$di \equiv (a; b, c) = a(b+c) - (ab+ac).$$

In commutative, associative, and distributive algebra it is respectively true that $co=0$, $as=0$, and $di=0$. But in nondistributive algebra the distributor is no longer equal to zero, since now ab or ac or both may be zero even though neither b, a, nor c is zero. This behavior of our distributor is analogous to that of commutators and associators, which are also not zero in $(-co)$ and $(-as)$ algebras.

Thus we have in N-algebra the possibility of two operators "zeroing out" or forming a mutually annihilating pair, agreeing with our basic $(-di)$ condition: $a \cdot b = 0$; $a, b \neq 0$. Such behavior should not be regarded as abnormal; on the contrary it is natural, and preserves the continuity of algebraic structure even after the breakdown of the norm-product rule, which extends only through Cayley algebra. The fact that the norm of a product should equal the product of the norms of the factors is intimately bound up with the representability of the product of two sums of n squares as the sum of n squares. This representability is in turn directly related to the possibility of continuing pure hypertetrahedral symmetry in higher spatial dimensions. Such symmetry extends only through eight dimensions, the representation space of C-algebra. With nondistribution we are entering spaces of nine or more dimensions, and in such spaces there is in general no longer a homogeneous or single-valued contact number (the number of spheres touching a given sphere) in lattices formed of equal tangent hyperspheres in such spaces. Also, in such spaces the hyperspheres of a D_{n-k} lattice section may not touch all those of the D_{n-k+1} section.

Mutual annihilation, which may be conceived of as the interference of two waves of π phase difference, finds its algebraic expression through disagreeing patterns of parity, each operator being characterized by such a pattern (see Section IV). In addition to the patterns of operators i_8 through i_{15}, given in the next section, patterns are also assignable to i_1 through i_7. Thus we have, where the numbers to the left signify the index,

1	0	4	$++$
2	$+$	5	$--$
3	$-$	6	$+-$
		7	$-+$

All nonfitting patterns "zero out" if in any binary factor pair there are three or more parity tracks in any one operator; $i_1 \equiv i$ fits with all patterns. Thus $i_3(-) \cdot i_4(++) = i_7(-+)$, but $i_3(-) \cdot i_8(+++) = 0$; whereas $i_3(-) \cdot i_{11}(-++) = i_{14}(+-+)$, for here there is no track annulling, since the minus in track 1 (the only track that i_3 possesses) agrees with the minus in track 1 of i_{11}. Thus also $i_7(-+) \cdot i_{15}(-+-) = -i_8(+++)$; but $i_7(-+) \cdot i_{14}(+-+) = 0$, and also $i_8 \cdot i_9 = 0 = i_8 \cdot i_{10}$, etc. Hence there are in

N-algebra $28+4(8)=60$ binary combinations of operators that produce zero, making 120 possibilities when anticommutation is considered.

There are structural rules governing the parity-pattern changes and assignments. But the basic table of Section II, showing multiplicative-operator subscript changes, together with the parity patterns given for each operator in this and the following sections, plus the basic rules for mutual annihilation already given, will enable the reader to work out any result without the extra labor of employing the parity-pattern structure rules.

Without entering into further detail here, one may summarize the interaction structure by the statement that a minus on any track of a factor reverses the parity of the same track of the following factor with more tracks, leaving all other tracks unchanged, whereas a plus sign on any such track preserves the parity of the corresponding track but changes the following tracks. Thus $(-)\cdot(-+)=(++)$; $(+)\cdot(-+)=(--)$; and $(++)\cdot(---)=(--+)$; and there are similar variations of the basic interaction rules for other parity patterns.

The operator $i\equiv i_1\equiv i_1(0)$ has the effect of a parity reversor; for example,

$$i_1(0)\cdot i_{10}(+--) = i_{11}(-++);$$

$$i_1(0)\cdot i_2(+) = i_3(-);$$

$$i_1(0)\cdot i_5(--) = -i_4(++),$$

etc.

If both operator and operand have the same number of tracks (with different patterns) then each track acts only on its correspondent; opposite parities yield (0), but like parities in the first track yield $(+)$, whereas in the second track $(-)\cdot(-)=(+)$ and $(+)\cdot(+)=(-)$. Thus

$$(++)\cdot(--)=(00)=(0)\equiv i_1;\quad (+-)\cdot(++)=(+0)=(+);$$

$$(--)\cdot(+-)=(0-)=(-);\quad (--)\cdot(-+)=(+0)=(+);$$

$$(-+)\cdot(++)=(0-)=(-),$$

this last parity equation referring to the operator equation in Cayley algebra, $i_7\cdot i_4=-i_3$ or $i_4\cdot i_7=i_3$. When simply the index of the resultant operator is considered, without reference to sign, then parity pattern multiplication can be regarded as commutative.

Similarly, the principal rule for the assignment of parity patterns is that in the case of an operator of even index the parity pattern is the compound one formed by adding together the patterns of the two factors of the index, the first being 2; and the parity pattern of the operator of next higher (i.e., odd) index is the polar or annihilating pattern for the preceding operator. Thus i_{14} is assigned $(+ - +)$ since $14 = 2 \cdot 7$, the patterns of the two factors being $(+)$ and $(- +)$, which combine to give $(+ - +)$. Then by the second rule the pattern for i_{15} is polar to the preceding, i.e., $(- + -)$. The ultimate validation for all these rules is that they optimally achieve a consistent algebra that embeds Cayley algebra and differs from it by the minimal number of structural and operational changes necessary to conform with the minimal type of nondistribution demanded by the breakdown of the modulus-product relationship in nine or more dimensions.

With algebras beyond N, which have operators of four and more tracks, there is a new possibility: $a \cdot 0 = b$; a, $b \neq 0$, ∞. This may be called *zero revival* and leads to the fact that $i_n \cdot i_m^{-1}$ may be equal to zero as well for $n,m \geqslant 16$. It is also true that our investigations show that viable (i.e., unique product) linear algebras are no longer possible in more than 128 dimensions. It is this phenomenon (which geometrically shows up as two or more sphere lattices with the same maximum contact number) which forces the appearance of *quadratic algebra* in a compound space of minimally 256 dimensions. At this stage a new type of number appears, characterized by $p^2 = 0$, $p \neq 0$. The writer discovered that for D_n, $n \geqslant 128$, these numbers must appear; and only later learned that Eduard Study had also deduced (from projective kinematics) the existence of such a number, although Study had no idea of the algebraic application or significance of his so-called "duale Nummer." From his method he could not know that the algebra demanding such numbers as $p^2 = 0$, $p \neq 0$, has a representation space of 128 dimensions and ends the viable linear algebras.

More will be said about quadratic algebra later, but this paper is concerned primarily with N-algebra, which has a quasi-isotropic representation space of 16-D rather than higher anisotropic algebraic spaces. Details of the still higher algebras we have found must await later occasions. N_C is the highest viable (nonschizoid) algebra permitting a finite number (128) of elements and the existence of a Laplacian operator, which, along with positive entropy, disappears in "cubic" algebra where a third kind of number, q, appears, with the characterizing equation $q^{\pm 4n} = \mathrm{RIP}(q_n) = 0$, where "RIP" means "real, imaginary, and dual (p) parts." In the multiplication of two powers of p (and *a fortiori* of q), the addition of exponents is not in general commutative. Both $p^0 = q^0 = 0$.

IV. The (*-di*) Elements of N-Algebra

1. $i_8(+++)$	5. $i_{12}(++-)$
2. $i_9(---)$	6. $i_{13}(--+)$
3. $i_{10}(+--)$	7. $i_{14}(+-+)$
4. $i_{11}(-++)$	8. $i_{15}(-+-)$

V. Beyond N-Algebra

A summary: $N_G \rightarrow N_H \rightarrow N_C \rightarrow N_N \equiv N^2$, i.e., quadratic algebra, after an isomeric transformation which involves the introduction of a new kind of concept: the *quadratic* operator implied by q This algebra has an infinity of elements and a morphology possessing relations with the elliptic modular function. These will not be discussed now. (There is also an absolute idempotent algebra to which one is naturally led by considerations stemming from the infinite-elements algebra above mentioned.) The two algebras beyond p involve q- and w-numbers (see pages 209 ff.).

VI. Commentary

This section, as explained in the Introduction, will be brief, and not exhaustive, dealing only with clarifications of certain salient points through the addition of further details.

The first such point concerns an apparently prevalent misconception pertaining to the nature of division algebras or those possessing unique factorization, that is, $(a \cdot b = 0) \Leftrightarrow (a \vee b = 0)$. C. S. Peirce corrected and simplified his father's unwieldy and arbitrary classification of algebras and first defined the concept of a division algebra. Charles Peirce also first concluded that R, G, and H were the only associative division algebras. Leonard Dickson in his otherwise very valuable work on algebra does not adequately refer to Peirce, and indeed commits the lapse in his "Algebras and Their Arithmetics" of saying that H is the last division algebra. The useful and valuable Condon and Odishaw's "Handbook of Physics" has in the last two editions repeated Dickson's lapse[2] but now as an apparent misconception—that quaternions "are the only hypercomplex system, apart from the reals and the complex numbers, which has no divisors of zero." This would mean that all division algebras are associative, which is not the case. Thus in Cayley or octave algebra if $\sum_{k=0}^{7} a_k i_k \cdot \sum_{k=0}^{7} b_k i_k = 0$, then $a_k = 0$ or $b_k = 0$; that is, there are unique divisors. W. W. Sawyer in his "Prelude to Mathematics" further repeats this repetition of Dickson's oversight, showing the present need for clarification of this point. That need is shown also by another omission in the handbook article: the absence there in the definition of Cayley algebra of the nonassociative multiplication and its rules, which form the distinctive and essential portion of the definition of that algebra.

The correct statement is that, although H is the highest associative division algebra, C is the last division algebra, as well as the last distributive algebra. After seven nonreal elements, the product of the norms no longer equals the norm of the product of two factors. The norm of such a product is then less, thus giving rise to nondistributive multiplication. Thus if N_1 and N_2 are two numbers in $A_N \equiv N$ (the first complete algebra after C-algebra), then if $N_1 \cdot N_2 = N_{12}$, $|N_1| \cdot |N_2| > |N_{12}|$. The dot in the equation refers to the (required) nondistributive multiplication of N-algebra, as defined in the foregoing tables, whereas the dot of the inequality refers to the ordinary multiplication of R-algebra.

Table I has shown how algebras are specifiable by their type of multiplication. In the two cases, R and G, indistinguishable by this criterion, the

[2] Dickson knew better, as his original discussion of quasi-quaternion treatment of C algebra in his "Linear Algebras" shows. A similar confusion arises in a 1963 paper (L. Inglestam, Hilbert Algebras with Identity, *Bull. Am. Math. Soc.* 69, p. 794)—that a division algebra must mean R, G, or H. However, division algebras include the nonassociative C-algebra. The same error of omitting C (all stemming from Dickson's ellipsis in "Algebras and their Arithmetics,") is made in a 1965 paper quoting Inglestam (M. F. Smiley, Real Hilbert Algebras with Identity, *Proc. Am. Math. Soc.* (p. 440)), the error in both cases escaping the referees and the editors of the Society. This is not a reflection implying incompetence, but a simple demonstration of how rushed information-processing has become, with inevitable losses in our societal memory.

additional and important distinction between amalgamative and nonamalgamative addition must also be taken into consideration. This distinction is defined by the fact that in R the sum of any two summands is expressible as a single (amalgamated) symbol, whereas in G this is not necessarily so, e.g., $(1+i)$ is not so expressible. The reason for nonamalgamation is metadimensional (see pages 209 ff.).

It is worth noting in passing that only the minimal changes of operational meaning are demanded by the successive algebras. Thus the noncommutation of H is simply anticommutation ($ab=-ba$), which is the very least change that could be made to render the multiplication noncommutative. Similarly, the $(-as)$ multiplication of C is not even mandatory except in certain specified triplets (i.e., those which are not $1\cdot 2=3$, $1\cdot 4=5$, $1\cdot 6=7$, $2\cdot 4=-6$, $2\cdot 5=7$, $3\cdot 4=7$, $3\cdot 5=6$, or any cyclic permutation thereof);[2a] and moreover, the nonassociative multiplication then resulting is again only minimally changed; i.e., it is merely antiassociative, that is, $a(bc)=-(ab)c$. If we consider a doublet as d and a singlet as s, $(-as)$ multiplication may be regarded as a meta form of $(-co)$, since $s\cdot d=-d\cdot s$.

The negative sign of any result in $(-co)$ may be interpreted as traversing the H-triplet in the opposite sense [see the diagram in Section II(b)]. Similarly, it may be shown that given the system of fixed singlet and doublet channels shown in Fig. 4, if we want to shift from flow pattern 1, $a(bc)$ to flow pattern 2, $(ab)c$, we must introduce a, b, and c into the channels formerly employed for c, b, and a, respectively.

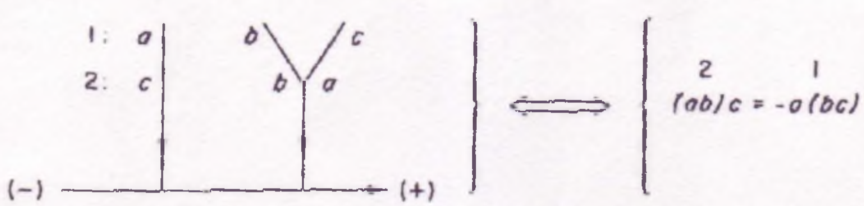


FIG. 4

[2a] Modulo-7 arithmetic (together with the rule that two even members in ascending order of value on the left-hand side will yield a negative result) directly generates the associative subscript equations of Cayley algebra. Similarly modulo-15 arithmetic generates the associative or H-triplets of N-algebra, in which, however, the additional complication of partially associative triplets exists, there being 35 H-triplets and 30 partially associative triplets (see the tables).

We have spoken of cyclic or H-triplets, which could also be called H-spheres. Each of these consists of three independent nonreal elements, thus forming an H-type of algebra. Now $i_{1,2,3} \equiv i, j, k$ may be considered as generating mutually perpendicular circuits (each containing the two real points ± 1) of unit radius such that the subscript equation $1 \cdot 2 = 3$ and its cyclic permutations governs the sense of all three rotations. (We noted this in a 1962 lecture at Naples, and it is discoverable in Hamilton actually—before Du Val's interesting book on quaternions and 3-space rotations appeared in 1964.) If we wish to remain entirely in D_3, it has not been generally noted, however, that the k-circuit must change size, since the i_3 or k-circuit must contain the points $+1 = k^0 = k^{4m}$ and $-1 = k^2 = k^{4m+2}$, where m is any integer. Thus the k-circuit expands, like a cone, with a base radius of zero (at k^0) to one of unity (at $k^1 = k$), then shrinks again to zero at $k^2 = -1$, the base plane of the cone moving so that its center traverses the real axis from $+1$ to -1 as $k^0 \to k^1 \to k^2$, the radius of k^1 being unity as already stated, and the slant height of the cone remaining unity, with vertex at the origin. At $k^{\pm 1}$ the altitude is zero, and the cone, a circle.

However, springing from the multivalued nature of $\sqrt{-1}$, H-algebra actually demands that $i_{1,2,3}$ be all on the same footing, which means a fixed size and position for all three circuits. If this requirement is thoroughly imposed, 3-space is no longer adequate, and we minimally need three fixed, mutually perpendicular unit circuits all intersecting in the points ± 1, which means a 4-space arrangement, such that only two i-circuits plus the real axis, i_0 (i.e., $i_{0,1,2}$, $i_{0,1,3}$, or $i_{0,2,3}$) could be represented in D_3 at any one moment. Thus quaternions do not exactly map 3-space rotations.

Since i_1 and i_2 imply $i_1 \cdot i_2 \neq i_1$ or i_2, we have, in our subscript notation, $(1,2) \to 3$. Similarly, $(1,2,3,4) \to (5,6,7)$ and $(i_k; k = 1,2,\ldots,8) \to i_{9,10,\ldots,15}$. Thus the (2^n)th i-element generates the algebra $A(2^{n+1})$ from $A(2)^n$. Thus $(i_0, i_1) \to$ G; (G, i_4) $\to$ C; and (C, i_8) $\to$ N, n being $2^{0,1,2,3,4}$, respectively, for R, G, H, C, and N, the number of i-elements in $A(2^n)$ being $2^n - 1$. This phenomenon is the basis of the notion of a complete algebra, which has been previously defined.

Since a D_4-sphere can have three mutually perpendicular great circles all intersecting a given pair of poles, any fully or cyclic (*as*) triplet may be called an H-triplet or H-sphere, each such H-sphere containing a quaternion algebra. N-algebra contains 35 H-algebras, just as C contains 7. The structure of N-algebra is extraordinarily richer and more complex than that of C, and far supersedes the richness of C with respect to H. There is a tremendous gap between the distributive and the nondistributive algebras.

It has not hitherto been realized that the norm-of-product/product-of-norms inequality characterizes $(-di)$ multiplication, nor that this in turn implies a volume-shrinkage relation among hyperspheres. Thus $\left(\sum_{k=0}^{m} a_k{}^2\right)^{1/2}$, where the a_k are all integers, is the modulus (positive square root of norm) of a number in $A(m)$, i.e., a linear algebra of m-elements. But it is also the radius of a D_m-sphere, as the D_m distance element proves.

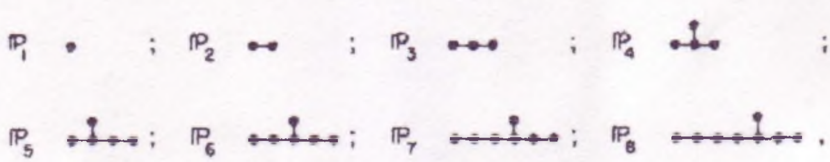
Thus the relation $\prod$ (norms) = norm($\prod$) that characterizes all distributive algebras means that $r_1{}^2 \cdot r_2{}^2 = r_{12}^2$ or $r_1 \cdot r_2 = r_{12}$, where r_1, r_2, and r_{12} are the radii of three D_m-spheres corresponding respectively to the two numbers and their product. Hence H-multiplication involves four-dimensional spheres, and C-multiplication involves spheres of eight dimensions.

From Table I and its footnotes, it is clear that the maximal number T_n of equal tangent hyperspheres that can fit about another central sphere of the same radius has an intimate connection with the number of units in a given linear algebra. It is hence of some interest to make brief mention of these tangent-sphere groups.

The writer has been interested in this problem since 1948, and approached it from the point of view of the convex polytope, which he called the *polytope of centers*, that might be formed by joining the centers of the shell spheres of a tangent sphere group. This led him to the discovery that contrary to the usual statement that the 24-celled regular polytope of D_4 has no ancestors and no descendants, it actually has both; for it is one of an infinite sequence of convex polytopes which the writer termed the hyper-cuboctahedra, since, like the cuboctahedron (the D_3 member of the family), all the other members also have a circumradius equal to any edge. This family is formed by the hypertruncation of the $(n-2)$th element of the hypercube. By this process the cuboctahedron, the 24-cell (the last regular, finite member of the family), and a 42-celled D_5 figure all arise as the representatives of this polytope sequence in three, four, and five dimensions, respectively. The family arises also from truncating hyperoctahedra.

What makes these three members most interesting is that they are also the packing polytopes ($\mathbf{P}_{3,4,5}$) or polytopes of centers for $D_{3,4,5}$. The author later found that the family of hypercuboctahedra could also be regarded as the ordinary linear truncations of the hyperoctahedra. The author then went on to define the more general family, $\mathbf{P}_n$, by the condition that (1) $_0r=e$; i.e., circumradius = edge, and (2) $N_0=\max$, i.e., the number of vertices to be a maximum, subject to the first condition. This is a much more fundamental sequence of polytopes than the hypercuboctahedra; and the writer learned that they had all been specified through $n=8$, although never realized to be members of one family. Indeed, although they had all been separately discovered by 1912, even in 1930 their connection with dense sphere packs was not realized, nor has the family been hitherto defined.

In a letter (May 1965), Professor H. S. M. Coxeter kindly noted that the packing polytopes' existence was implied by his 1951 paper in the *Canadian Journal of Mathematics*, p. 414, although not realized as such or defined by a family. In Coxeter's valuable graphic notation the members from $n=1$ through $n=8$ can be written



a notation which can be regarded as mapping whole dimensions into points, as does (in a different manner and independently arrived at) the writer's archemorphic notation for classifying differential equations (*17*, p. 256). That $N_0(\mathbf{P}_8)=240$ was first discovered by the genius of Thorold Gossett in 1897, although the connection of the number 240 with the eight-dimensional tangent sphere group (T_8) was not realized until H. F. Blichfeldt's brilliant work on minimal quadratic forms in 1935; and even thereafter the geometric connection with Gossett's work was slow in coming, Coxeter being the first to realize it in an insightful paper (1946) which linked that number also to C-algebra.

In 1963 the writer found the following results, the summary formula of which is quoted in (*8*, p. 66).

Define T_n as the maximal number of equal tangent hyperspheres that can all touch another sphere of the same radius in n-dimensional parabolic space, D_n. Then for $1 \leqslant n \leqslant 8$,

$$T_n = n\left(\left\{\frac{2^{n-2}}{3}\right\}+n\right),$$

where $\{m\}$ is the least integer containing m.

Coxeter ("Regular Polytopes," 2nd ed., 1965, p. 234) requires five (the last three being quite complicated) separately derived equations and expressions to gain h, the period of the product (order of multiplication irrelevant) of the finite symmetry group generated by reflections of a hypertetrahedral fundamental region. These groups are extremely important, are all irreducible, and can extend only up to eight dimensions, inclusively. After $n=8$, the fundamental region is reducible or factorable, and there are no more such groups. The graph is then disconnected.

The expression (in the above formula) multiplying n is h for these groups, thus furnishing the periods of the products of their generators in remarkably uniform, concise, and simple form using one equation only. The geometric basis of our expression for T_n extends very deeply into the fundamental nature of irreducible groups generated by reflections. The expression in brackets contains the heart of the matter, and represents the maximal number plus one[3] of hypertetrahedral vertices (i.e., the kind with three cells, all also of hypertetrahedral shape, about a vertex) that can be formed given 2^{n-2} hypertetrahedral cells. That number of cells in turn is the number of cells bounding the hyperoctahedron in $(n-2)$ dimensions.[4] Thus the

[3] The extra "1" refers to the $n(n+1)$ vertices always formable in n dimensions by translating the $(n+1)$ cells of an n-dimensional hypertetrahedron outwardly along their respective altitudes (drawn from the center of the figure to the centers of each cell).

[4] The regular D_n octahedron has the symmetry of a hypersphere in D_n with all its $(n-2)$ mutually perpendicular axes of rotation drawn with respect to a given equatorial plane. The $(2n-4)$ poles of such a sphere have the symmetry of the $2(n-2)$ vertices of an $(n-2)$-dimensional octahedron, since the n diagonals of a regular D_n octahedron are all mutually perpendicular. In the prior equation, $\frac{1}{2}T_n$—the number of diameters in T_n—is the number of hyperplanes of symmetry in the irreducible group of period $h=T_n/n$ generated by reflections. The fundamental region for such a group is always a hypertetrahedron in elliptic or parabolic space, and such groups extend only through $n=8$, as mentioned above.

previous equation by no means simply epitomizes existing information on these groups but provides a new and fundamental insight into the nature of finite groups generated by reflections, as well as a demonstration of the unity of the first eight dimensions in concise, elegant fashion.

From our formula, $N_0(\mathbf{P}_n)=2, 6, 12, 24, 40, 72, 126, 240$ for $1 \leqslant n \leqslant 8$, which are also key numbers in the theory of quadratic forms, sphere packs, and tetrahedral symmetry groups. It is a consequence of their definition that in all these convex polytopes the number of edges meeting at each vertex is the same in each such polytope; i.e., $2N_1/N_0=N_{1/0}$. Also all the dihedral angles of such a polytope are equal, the dihedral being defined as the angle between two D_{n-1} elements. These facts in turn imply that each vertex can be surrounded by not more than two kinds of surface cells c_1 and c_2, so arranged that in the hypersurface pattern all c_1's are adjoined at all D_{n-2} elements by c_2's, and vice versa. For $n=2$ or 4, $c_1=c_2$, and we have all α_1's and all β_3's, respectively, as surface cells, that is, a regular figure; in a more trivial sense, $n=1$ also produces a regular figure. After $n=4$ this phenomenon can never occur again in any finite positive dimension, and in all other cases $\mathbf{P}_n$ must be a special type of Archimedean figure such that if ϕ be the angle at the center between any two vertices, $\phi=\pi/3$ radians.

After $n=8$ tremendous changes occur, in line with the vast shift from distributive to nondistributive algebras. Put in terms of the theory of sphere packs, the polytope of centers is no longer convex.[5] The writer's findings (*17*, pp. 230 and 262) for tangent sphere groups with $n \geqslant 9$ refer to the maximal numbers of equal tangent spheres surrounding another of greater radius such that the difference between the two radii is minimal, since the two radii can no longer be equal and the polytope remain convex. The writer's previous work was done under the assumption of a convex

[5] Note that the edges of a "polytope of centers," and hence of a "packing polytope" ($\mathbf{P}_n$), are all lines of centers between pairs of *tangent* (hyper)spheres in the T_n (hyper)shell. A packing polytope, as will appear from the exposition, is simply a maximal-vertex (N_0 max) polytope of centers for a monoradial (hyper)sphere pack in a given D_n, "monoradial" referring to the fact that both the shell (hyper)spheres and the central one all have the same radius. When the radius of the central member is different from that of the shell members, all of which are alike, we have a biradial pack, also considered in pages 192 ff. The packing polytopes summarize in their structure the sphere-packing or lattice possibilities of their dimensions.

polytope of centers as his theorem that for $n > 8$, the polytope of centers is no longer convex, was only recently found. The proof of this theorem in quickest form may be made to devolve upon the equation

$$4 \csc^{-1} \sqrt{n} - \sec^{-1} n = 0,$$

which has as its only real solution $n = 8$.

This equation represents the last finite member of our *alphabet honeycombs*, a remarkable series of hyperspace lattices, all the cells of which consist only of alphas and betas, i.e., of regular hypertetrahedra and hyperoctahedra; hence our designation "alphabet." Where the following coefficients represent numbers of dihedral angles of the indicated polytopes, we have for the six finite members of the series (all the several expressions in parentheses being equal to 2π radians), in D_2, D_3, D_4, and D_8:

$$(6\alpha_2) = (3\alpha_2 + 2\beta_2) = (4\beta_2) = (2\alpha_3 + 2\beta_3) = (3\beta_4) = (1\alpha_8 + 2\beta_8).$$

There is also a seventh member belonging to D_∞ $(2\alpha_\infty + 1\beta_\infty = 2\pi)$, since the dihedrals are now, respectively, $\pi/2$ and π.

The reason for the great symmetry and density of the packing structures in D_4 and D_8 is intimately connected with the following facts, the significance of which has not been noted: the circumradius of the D_4 cube (γ_4) is equal to its edge, and in γ_8 it is equal to the face diagonal. Thus the basic symmetry possibilities of the cube are fully exploited at $n = 8$ in so far as single straight lines can be related. There is one simple symmetry possibility, but even this involves two perpendicular straight lines related to a third line; the circumradius of γ_{16} is equal to the sum of the two sides of the isosceles right triangle whose hypotenuse is the face diagonal. This D_{16} relationship breaks the simplicity of the single linear pattern, of which D_8 is the last representative; however, D_{16} is still simple enough to be the highest dimension in which a simple $(-di)$ algebra may exist, i.e., a nondistributive algebra without the complication of zero revival, that is, without $a \cdot 0 = b$, where $a \cdot b \neq 0, \infty$, but with $a \cdot b = 0$.

There is not space here to enter into the indicated processes, which rest upon derivable rules of zero formation. Suffice it to say that six levels of zero arise through the interaction of nondistributive operators in N through N_N algebra, these six falling into a hierarchy of three parity pairs. Hence six kinds of zeros result, which are indicated by the following table, showing alternative symbols. These have relevance in quadratic algebra.

TABLE II

THE SIX ZEROS OF (*-di*) MULTIPLICATION (cf. Sections III and IV)

1	$(+)(-)$	$\equiv (\pm)$	$\equiv \left(\begin{smallmatrix}\uparrow\\ \downarrow\end{smallmatrix}\right)$	$\equiv 0$	
2	$(-)(+)$	$\equiv (\mp)$	$\equiv \left(\begin{smallmatrix}\downarrow\\ \uparrow\end{smallmatrix}\right)$	$\equiv \bar{0}$	
3	$(0\cdot\bar{0})$	$\equiv \binom{0}{\bar{0}}$	$\equiv \left(\begin{smallmatrix}\leftarrow & \rightarrow\\ \rightarrow & \leftarrow\end{smallmatrix}\right)$	$\equiv \supset\subset$	$\equiv \odot$
4	$(\bar{0}\cdot 0)$	$\equiv \binom{\bar{0}}{0}$	$\equiv \left(\begin{smallmatrix}\rightarrow & \leftarrow\\ \leftarrow & \rightarrow\end{smallmatrix}\right)$	$\equiv \supset\subset$	$\equiv \bar{\odot}$
5	$(\odot\bar{\odot})$	$\equiv \binom{\odot}{\bar{\odot}}$	$\equiv \left(\begin{smallmatrix}\supset\subset\\ \supset\subset\end{smallmatrix}\right)$	$\equiv \begin{smallmatrix}O \mid O\\ O \mid O\end{smallmatrix}$	$\equiv \oplus$
6	$(\bar{\odot}\odot)$	$\equiv \binom{\bar{\odot}}{\odot}$	$\equiv \left(\begin{smallmatrix}\supset\subset\\ \supset\subset\end{smallmatrix}\right)$	$\equiv \begin{smallmatrix}O \mid O\\ O \mid O\end{smallmatrix}$	$\equiv \bar{\oplus}$

Thus $(+--)(-++)\to(00\bar{0})$; $(0\bar{0}0)(\bar{0}00)\to(\odot\bar{\odot}\odot)$; $(\bar{\odot}\odot\bar{\odot})(\odot\bar{\odot}\odot)\to(\bar{\oplus}\oplus\bar{\oplus})$, etc. It should be noted that $\oplus$ and $\bar{\oplus}$ do not interact, each containing a self-polar system of rotations as the fourth symbols of rows 5 and 6 show. In N_C-algebra is thus attained the highest level zero of the hierarchy, i.e., $\oplus$ or $\bar{\oplus}$ on each of six "tracks" or channels (a "6-track zero"), the $(-di)$ operators in that algebra being able to have as many as six channels or guides for their parity waves; whereas N, N_G, and N_H may have only 3, 4, and 5 parity wave channels respectively. Six channels is a limit that may be compared to a coaxial cable formed by six helical strands around a core of equal diameter. There is not space for further development here, but class numbers of higher cyclotomic number fields are relevant.

Although nondistributive multiplication and its possible algebras have not hitherto been considered, they constitute as fundamental and basic an extension of mathematical thinking as the extension from positive to negative, and then to complex and hypercomplex numbers. What has delayed such specification was the unreasonable fear of a non-division algebra, that fear being no more tenable than it was in number theory, where Kummer freed us from confinement to algebraic number fields with unique factorization in terms of real or complex numbers.

The same problem arose in vector algebra. Thus if v and v' are two perpendicular vectors, and v_1 is another vector not perpendicular to either, we have for the scalar product $v \cdot v_1 = v_2$, but $v(v' + v_1) = v_2$, since the scalar product of two perpendicular vectors vanishes. Thus nondistribution may arise in vector algebra, rendering vector quotients nonunique, since any other vector v_p, perpendicular to v, would yield $v \cdot v_p = 0$.

On another occasion (*17*, p. 216) we have also shown that $(-di)$ multiplication enters into the theory of nonlinear operators, with the consequence in electronics that undistorted modulation is not possible with a nonlinear operator, and neither are superposition or classical harmonic analysis; and we add *en passant* that distorted modulation is deeply connected with any theory of mathematical esthetics, a field that has interested mathematicians from ancient times, notably G. Birkhoff in our times, although Birkhoff did not reach the connection between nonlinearity and esthetics. In this connection the volumes brilliantly edited by G. Kepes of the Massachusetts Institute of Technology (published by Brazilier, New York) on structure and the visual arts should also be noted, even though not quite relevant to the present technical bibliography. They reached the writer's attention after this paper was in press and hence could receive only brief mention. [See also (*17*, p. 227 ff.) and pages 200 ff. of this paper for relations to biological patterns and coding structures.]

Professors Antoniewicz, Conti, and Moreau have, each in their own way, stressed the relation of convexity to control. We are suggesting that this relation can be made more precise, analytic, and applicable by the following *Theorem*: The *unit packing structure* of a parabolic dimension n governs the controllability problem in such a space. *Lemma*: That packing structure is in turn specified by the maximum number T_n of equal tangent hyperspheres in D_n that fit about another of equal radius; then $T_n = N_0(\mathbf{P}_n)$, defining the *polytope of centers* by its number of vertices; that all its edges are equal follows also.

There is a natural hyperplane separation of these sphere groups (cf. Conti's "ball of controllability") which we characterize thus: two "polar caps," each of $\frac{1}{2}(T_n - T_{n-1})$ spheres, are separated by an $(n-1)$-hyperplane of "equator" T_{n-1} sphere units.

Concluding our previous remarks on sphere packs, we note that it is therefore now possible by means of the concept of the *polytope of centers* to unify the geometry of all sphere-inscribable polytopes of equal edge in terms of rigid groups of tangent spheres (radius r) about a central one of radius R. If ${}_0R$ is the circumradius of such a polytope, whose edge is e, then $[2({}_0R)/e]-1 = R/r$. Thus R/r determines the polytope. If $R<r$, we have all the regular figures (except γ_4 for which $R=r$); and for $n<8$, the Gossett polytopes. If $R=r$ we have all the packing polytopes $\mathbf{P}_n$, all the truncated cross polytopes (it is interesting to note that $t\beta_n = \mathbf{P}_n$ for $n=3$, 4, 5), and all the expanded simplexes. For other ratios of R and r we have the truncated expanded simplexes, and the hemifigures of these truncations, all those ratios being of the form R/r, as are all γ_n for $n>4$. The hemigammas for $n>8$ have $R>r$; for the hemigamma $n=8$, $R=r$; and for $n<8$, $R<r$. The 240 vertices of $\mathbf{P}_8$ are thus obtainable by compounding a $h\gamma_8$ and a truncated β_8 $(128+112=240)$; this is believed to be a new construction.

Thus all the Archimedean figures are subsumed under the theory of rigid configurations of equal spheres on a central sphere of usually larger radius.

The ratio $R/r = \sqrt{5}$ is particularly interesting. In D_2 it yields the decagon, in D_3 the icosidodecahedron, and in D_4 the 120 cell as well as a figure of 96 vertices (see below). It appears to be the only finite ratio other than unity which generates polytopes that, like the packing polytopes, possess analogous sections in lower dimensions which are all maximal-contact packs for that ratio. Thus with $R/r = \sqrt{5}$, no more than 10 spheres fit about 1 in D_2; no more than 30 in three dimensions; and no more than 120 in D_4. In five or more dimensions, the ratio no longer yields a rigid pack.

In four dimensions, rigid packs (for $R \geqslant r$) can be formed from 16, 20, 24, 32, 96, 600, or 720 equal spheres. It is interesting to note that the group of 96 has also the ratio $R/r = \sqrt{5}$. Similar theorems may be enunciated for the higher dimensions with this theory. In all dimensions $(n>2)$ the number of rigid packs is *finite*, as is the number of viable ratios R/r.

Thus the subject of generalized sphere packs, as here outlined, has an intimate connection with the theory of polytopes. The group 360-about-1 in D_9 is the "first" convex figure in that it is the first convex pack in D_9 with the number of contacts to the central sphere exceeding 272 and with $(R/r)-1$ a minimum. This group forms a polytope of centers that could be called a "$hte\alpha_9$", i.e., a hemitruncated, expanded D_9 simplex, using the notation introduced by Coxeter in his 1930 paper.

One can go further and treat the theory of all equal-edged convex polytopes, whether sphere-inscribable or not, in terms of maximally dense packs of equal spheres, using the notion of a polytope of centers. Similarly, equal-edged nonconvex polytopes may also be treated and generated, but there is no room here for the interesting details. We shall hereforth adopt the names *alpha*, *beta*, *gamma* as more convenient than the relatively awkward "simplex" or (hyper)tetrahedron, "crosspolytope" or (hyper)-octahedron, and "measure polytope" or (hyper)cube, respectively. Indeed, "simplex" is somewhat of a misnomer. For the (hyper)tetrahedrons are one of the subtlest and most elegant of all polytopes, ensuring as we have seen they do that the eighth dimension is the principal key to the theory of structure.

Related to this fact is the writer's expression $(\Delta_g)^{1/2}/g$ ($g=9-n$, n referring to D_n, and Δ_n being the nth triangular number, where $\Delta_1=1$, $\Delta_2=3$, $\Delta_3=6$, etc.) for the governing structural ratio of circumradius to edge in the dimensionally successive vertex-truncation figures of $\mathbf{P}_n$ ($n=9$ yields a lattice or infinite polytope). This series of polytopes was first discovered by the geometrical genius of Thorold Gossett in late 19th century England, and called by him "the wedges"; although he did not recognize the series as one of successive vertex figures nor as one related to the highest possible convex polytope of monoradial centers, our $\mathbf{P}_8$.

During an interesting and stimulating conversation in London after this section had been written, but in time to include this comment, Professor C. A. Rogers kindly brought to the writer's attention his recent paper (*Mathematika*, 1963) on covering a sphere with spheres. Here Rogers, despite excellent results, reaches an impasse of sorts in considering a problem which in terms of the present theory reduces to that of a polytope of centers with $R>r=1$.

On p. 157 Rogers states frankly that "the following results ... are not completely satisfactory." Thus when $R>n\log n$, Rogers concludes, "I do not see how to obtain a really satisfactory form of theorem 1 in this case."

The present work suggests that the initial assumption $R > r$, $r = 1$, was too restrictive, and moreover, that consideration of the ratio R/r, and not R or r considered separately, unlocks the algebraic geometric morphology of this problem and situation. The method and notion of the polytope of centers is likewise relevant, as well as the important theorem that for $n \geq 9$, $\mathbf{P}_n$ is nonconvex. There are two other important considerations in this same problem which are directly connected to the situation for large n. The first is an interesting theorem of the writer resting both on the Coxeter-Rogers-Schläfli upper bound and on the writer's formulas for sphere packs with convex polytope of centers:

$$\lim_{n \to \infty} T_{n+1}/T_n = \sqrt{2}.$$

The second consideration referred to above grows out of Rogers' fundamental result of 1958 (*Proc. London Math. Soc.*) based upon findings by the great Ludwig Schläfli. Rogers then showed that the volume ratio of the part of α_n taken up by $(n+1)$ tangent hyperspheres with their centers at the hypertetrahedron's vertices to the whole volume of α_n is, asymptotically, $\sigma_n = (n/e)2^{-n/2}$, i.e., $\lim_{n \to \infty} \sigma_n = 0$, more simply, $\sigma_\infty = 0$, a result not explicit in Rogers' paper.

The empty portion of the simplex, i.e., the n-dimensional hole, may then be calculated to be asymptotically given by

$$H_n = \frac{(n-1)^{1/2}}{n}\left(2^{n/2} - \frac{n}{e}\right).$$

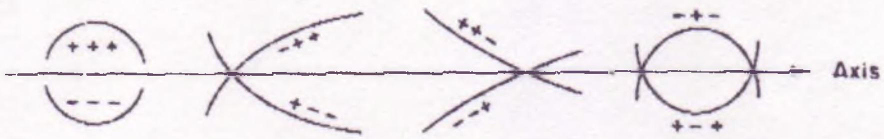
Hence

$$\frac{\text{empty portion}}{\text{filled portion}} = \frac{1}{\sigma_n} - 1 = \left(\frac{e}{n} \cdot 2^{n/2} - 1\right).$$

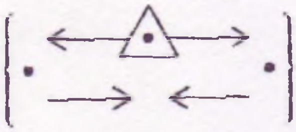
Thus our theorem—that for large n the empty portion increases at the expense of the filled portion in the ratio $(e/n)2^{n/2}$, and that although the D_n content of α_n approaches zero as $n \to \infty$, the portion of α_n filled by the $(n+1)$ hyperspheres becomes progressively less with regard to the empty space, and this ratio approaches zero as $n \to \infty$.

Therefore the unusable surface of a D_n sphere, with regard to a maximal number of equal spheres tangent to its surface, increases half-exponentially with n. It is the lack of taking into account this unusable part of the hypersurface that makes the Schläfli-Rogers-Coxeter upper bound too high. That that bound is valid as far as it goes, however, has been demonstrated by the writer from the fact that the vertex-content ratio is greater for the alpha than for any other regular or partially regular figures. This result was communicated to Professor Coxeter in October 1965, about a month after it was obtained.

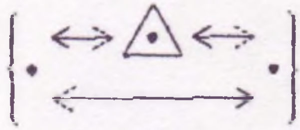
Before closing this section of commentary, a few points should still be mentioned, such as that on page 176 the table of the 35 H-triplets has been deliberately "scrambled" to provide an exercise for the reader interested in gaining more familiarity with the indicial equations of N-algebra. By the use of modulo-15 arithmetic in the subscript equations, the table may be arranged in perfect sequential order. (*Hint*: Use the type of subscript equation $n \cdot m = r$ or $-r$ with $n \leqslant m$ and both n, m odd or even, respectively, where maximum r is the sum modulo 15 of n and m or a lower number determined by selection rules necessary for self-consistency.) This table involves 35 separate, but related, H-type algebras and one C-algebra. Consistency with both the C and H algebras embedded in N serves to derive the multiplication table. The eight $(-di)$ operators arise from the simplest type of parity wave oscillating about a neutral axis; namely, the eight permutations of two things (+ and −) taken three at a time, for three moments or points, are minimally required to establish oscillation or curvature. The eight forms fall into four pairs (cf. Section IV):



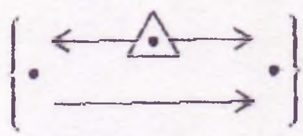
It should also be noted that the table in Section II(c) exhibits the *partially* cyclic triplets whose linearly projected flow patterns (△ denotes vertex of triangle) are



instead of



as in the (fully) cyclic or H-triplets, thus allowing two categories of non-associative multiplication by means of two triplet types that may thus be termed hyperbolic and elliptic, respectively. The *linear* "triplet" of Section II(a) can in the same spirit be termed parabolic, possessing the vertex pattern



The three patterns may be designated as hyperbolic or self-opposing; elliptic or self-affirming; and parabolic, equilibrated, or mixed, respectively.

Similarly, R-algebra, where, u being a unit, $u^2=1$, may be termed "elliptic"; all the complex and hypercomplex algebras, wherein $u^2=-1$ are thus "hyperbolic"; and the quadratic algebra to which $\mathrm{N_N}$ leads includes $u^2=0$, i.e., "parabolic." There is a fourth category of unit, and hence of number, in the algebra with an infinity of elements where, although $u^4=0$, $|u|=1$. This last category may be called "loxodromic." This terminology is derivable from isomorphisms to linear transformations or conformal representations, and the traces of their matrices, the matrix being loxodromic if the trace ($\equiv$ sum of elements of principal diagonal from upper left to lower right) is not real.

We have already noted that every matrix represents a hypercomplex number. In general i_n has a (2^{m-1})-element real matrix representation in all algebras $A(2^m)$, $m \geqslant n$. Thus $i_1 \equiv i$ requires a 4-element real matrix, whereas $i_3 \equiv k$ or $i_2 \equiv j$ requires a 16-element real matrix. Thus, for example,

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \leftrightarrow i_2.$$

(Another matrix identical with i_2 is similar but smaller, containing $\pm i$ as well as ± 1 and 0.) It is not difficult to decide which notation is most elegant and least cumbersome. Thus hypercomplex numbers can be regarded as embedding in themselves entire systems of linear equations. The principal diagonal (upper left to lower right) of all real-element matrices referring to hypercomplex numbers is null.

The author found that $p^2 = 0$ was demanded when hypercomplex numbers attain a surdimensionality of 8, the *m*th surdimension being defined as a higher hierarchical form of a (2^m)-dimensional parabolic or Euclidean space. These surdimensions are very important in defining what we have called "complete algebras." In terms of them, real, Gaussian, quaternion, Cayleyan, and N-algebra are 0-, 1-, 2-, 3-, and 4-surdimensional, respectively. Beyond 4 surdimensions, as already indicated, there is a hierarchy of zero revival in successive complete algebras, until finally $p^2 = 0$, $p \neq 0$, is attained, in 7 surdimensions. At this point linear algebras give way to quadratic algebra and then cubic algebra (see Section IV). There is reason to regard these surdimensions as *logical closures* with respect to hierarchies of operational structures composed of independent categories (cf. (*18*), Addendum).

Beyond the first $(-di)$ algebra (i.e., N) there lie at least two higher forms of nondistribution, which may be symbolized as providing the results $a \cdot 0 = b$ and $a^2 = 0$, respectively, where neither a nor b is zero or infinity. It was with great delight that, after having confirmed the necessity for $p^2 = 0$, $p \neq 0$, in higher $(-di)$ algebra,[6] we found that Eduard Study had in 1900 arrived at the same type of number from considerations of nonreal projective geometry, arising originally out of his new approach to kinematics. These numbers, called *dual numbers*, have been regarded more or less as a mathematical curiosity, with no inkling of their transcendent importance in the theory of nondistributive algebraic structures.

The simplest kind of number characterized by the fact that the square of the unit is minus unity, is termed an imaginary number and the form of its

[6] Specifically in N_C, the algebra of metacomplex numbers of the p-type with 2^7 elements, i.e., the 7th surdimension or algebraic field, all fields beyond the 3rd surdimension being nondistributive.

unit power field is a unit circle on the complex plane. Similarly, the hyperimaginary numbers (i_n, $1 \leqslant n$) form a series of hyperplanes, the entire series constituting what we term the hypercomplex plane. The kind of number characterized by $p^2 = 0$ where p is the unit, is related not to simple circles, but to a pair of tangent circles of unit diameters. There is a relation here to the complex function $w = z^{-n}$, which yields a family of tangent circle pairs for $n = 1$. In Cartesian coordinates one such pair, representing the unit field form of this second kind of higher number, is given by $(x^2 + y^2)^2/y^2 = 1$, the radius vector for an angle of radians from the real axis being given by $r = \sin\theta$, and hence $p^{\phi} = p^{2\theta/\pi} r\ (1 - r^2)^{1/2} + pr^2 = \sin\theta(\cos\theta + p\sin\theta)$. Thus $p^0 = 0$ and $p^2 = 0$, which distinguishes p- from i-numbers.

Just as all the i_n-operators determine the hypercomplex "plane," i.e., a hyperplane of n nonreal independent lines and one real line, so the p_n numbers determine a metacomplex "space," the rotation planes of all the p_n being perpendicular to the hypercomplex plane. The unit field form of an i_n-operator is a unit circle, and that of a p_n-operator, two tangent circles of unit diameter. In plane perspective hypercomplex space can be represented as a circle-ellipse family, metacomplex space as a double circle-double ellipse family.

These p numbers, which we arrived at through a theory of algebraic structure, and which Study arrived at through a kinematic, projective analysis inspired by some work of Laguerre, have, as J. Grünwald pointed out about 1906, an isomorphism with the quadric cone in Study's scheme, although we have found the double circle family, derivable from the complex function already given, more useful and accurate for our purpose. Actually, the two views can be reconciled when the double circle curves are regarded as the vertical sections of an infinite sequence of what we have previously termed umbilicoidal shells in a 3-space. These umbilicoids are intimately related to the even negative dimensions[7] and to our theory of

[7] A simple umbilicoid is given by $(x^2 + y^2 + z^2)^2/(x^2 + y^2) = 1$. The content of odd negadimensional umbilicoids is zero, just as that of even negadimensional spheres is zero. Conversely, umbilicoids exist in the even negadimensions, whereas spheres do not. The two are in this sense complementary forms. The (hyper)surface of an n-dimensional umbilicoid is given by

$$2(n-1)\,\pi^{(n+1)/2} \frac{r^{n-1}}{[(n-1)/2!]}.$$

half-integer genus, with corresponding Riemann-surface representation. As the over-all horizontal diameter of these shells tends toward infinity, the finite portion in the neighborhood around the origin tends to a quadric cone, much in the same way, as we have previously pointed out (*17*, p. 258, note), as a finite portion of an infinite elliptical torus may tend toward a hyperbolic paraboloid.

It is interesting to note here that hyperspirals may be defined as projections on a D_{n-1} hyperplane of hyperconical helices in D_n. Thus the complex numbers may be ordered in terms of an Archimedean spiral of infinitesimal pitch; and it is just as incorrect to state that complex numbers cannot be ordered as that the points on such a spiral line cannot be ordered, radius vector and angle being given first and second priority, respectively, thus generating an infinity of numbers, all ordered for each member of a like infinity of radii, also all ordered.

The hypercomplex numbers can likewise be ordered by the hyperspirals we have defined above, with a hierarchy of priorities assigned to radii and angles in successive dimensions. Professor Charles Loewner, in his invited lecture before the American Mathematical Society in 1964, came close to such an ordering, but reached only the stage of hypercones, and thus

where n is the dimension (n may be negative) and r, the radius of the two generating hyperspheres. Similarly, the content of an n-dimensional umbilicoid is given by

$$\frac{2\pi^{(n+1)/2} r^n}{[(n-1)/2]!}$$

Thus, when $r=1$, $n=3$, we obtain a surface of $4\pi^2 r^2$ and a volume of $2\pi^2 r^3$, tallying with the equation given above. The negative dimensions are deeply related to the nature of frequency, and hence, of time, rather than space. They have not been hitherto considered. D_{-1} is by far the most important of all negative dimensions. In this connection, by use of the Riemann zeta function we have obtained the result that

$$\lim_{\epsilon \to 0} (-1+\epsilon)! = S_\infty$$

where $\epsilon \geq 0$ and $S_\infty \equiv 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$, the famous harmonic series. Likewise $(-2)! = -(-1)!$ and $(-3)! = \frac{1}{2}(-1)! = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$. Thus the first three negadimensions are the most important, where n is any positive integer.

$$(-n)! = (-1)^{n-1} S_\infty/(n-1)! \text{ and } (-1)!/(-n)! = (-1)^{n-1}(n-1)!$$

Thus D_{-1} enters into the nature of Euler's constant, as $n!$ and D_n are closely morphologically related. That $(-\infty)! = 0$ follows also.

attained an admitted ambiguity, which the concept of the hyperspiral eliminates. We so advised Loewner in 1964.

Finally, related to the sixth-order curve $x^4 = y^4 - y^6$ and a family based upon it through an appropriate parameter, we have the number q_n such that $|q_n| = 1$, $q_n{}^{\pm 4k} = 0$, which do not participate in any of the number spaces determined by the i- and p-numbers, but participate only in algebraic structures beyond quadratic algebra (see Section V and pages 192 ff.).

VII. Application to Error-Correcting Codes

There is a deep connection between dense-packed, error-correcting codes and maximal monoradial groupings of tangent hyperspheres about a central hypersphere. The Golay-Paige code can be derived directly from such a maximal pack in 23 dimensions, the partition of the exponent being $11 + 12$, in the sense that the 23-dimensional cube may be considered a pattern of 2^{11} vertices repeated 2^{12} times. This pattern is related directly to sphere packs in 23 and 24 dimensions, as Leech (*14*, pp. 670–671) has shown. The reader is also refered to Refs. *15* and *19*.

Higher dimensional packs suggest that still more sophisticated codes and hence computer programs would arise from packs of tangent hyperspheres with more than one lattice possible.

During a conversation at the University of London, Prof. C. A. Rogers brought to attention a very interesting paper of his (*Mathematika*, 1957), following discussions with Dr. S. K. Zaremba on the remarkable efficiency of a random redundant code in transmitting through an imperfect channel. In this paper Rogers sums up the findings, including his own and those of Professors Bambah, Davenport, Roth, and Watson, to the effect that the most economical lattice covering of a space by congruent figures whose centroids form the points of the lattice are nevertheless less economical than the most economical covering possible, which Rogers showed has an upper-bound density of $n(\log n + \log\log n + 5)$ for $n \geqslant 3$, n being the dimensionality of the space.

We first venture to say that the efficiency is due to the redundancy of the code rather than its randomness, which would, however, increase Shannon information, just as a nonsense message has more unexpectedness or more such "information" than a meaningful message. This observation leads to the interesting theorem that a cipher message originally composed in some language has less Shannon information than a message of nonsense syllables; i.e., any cipher must preserve some form of the original pattern of thought, and hence less unexpectedness than complete nonsense.

The principal reason for mentioning these facts is that a random covering may be more economical than the most economical *one-rule* lattice covering. We have introduced the italicized adjective, feeling that it lies at the root of the apparent paradox, for random coverings of maximal efficiency could be defined only by *shifting* rules of formation. It is this more flexible strategy, as it were, that accounts for their possibility of being more efficient than even the most economical lattice covering, which is based on a *single* rule of formation.

VIII. Other Applications

Aside from the applications to geometry and computer coding and programming already mentioned, there are important relations both to the theory of numbers and to physics.

We shall mention the first very briefly. As long ago as 1880 Henri Poincaré had very masterfully pointed out that the ellipse and the hyperbola were the geometric keys to the structure of quadratic number fields based on $\sqrt{-K}$ and $\sqrt{K}$, respectively. It was also known by the late 19th century that the ideals of a quadratic algebraic number field (of multiple factorization) were isomorphic to lattices of constant mesh area. In the simplest cases such lattices are planar, thus suggesting at once the machinery of elliptic functions. In more complicated cases they may exist on specially defined surfaces which the writer has found are intimately related to his "curvilinear elliptic functions," which may be defined very briefly as more generalized elliptic functions based on curvilinear parallelograms. There is an extension to higher (than quadratic) number fields and their corresponding higher-space lattices, both flat and curvilinear, for which we have no space here, except to say that multiperiodic and hypercomplex functions are involved.

Such a development relates the theory of algebraic number fields to packings of hyperspheres, the centers of which would be the lattice points; just as we previously saw that such packings were intimately connected with the theory of hypercomplex numbers and hence with the structure of algebras.

The matter goes even deeper, since matrices may be treated, often with great gain in succinctness and elegance, as hypercomplex numbers. Moreover, the theory of finite groups in its most recondite aspects is related to the notions of a group of tangent hyperspheres and its polytope of centers. We have already seen the connections with error-correcting codes.

Thus groups of tangent hyperspheres, determining as they also do the packing structure of any given dimension, constitute one of the most fertile and fundamental domains of mathematics. Algebraically, hypercomplex numbers possess the same fundamentality, determining as they do even the nature of the arithmetic operations that may be performed upon them or, more accurately, in which they may be engaged. For the nature of a kind of number determines the nature of its operations. Since mathematics itself may be defined as the science of numbers[8] and their operations, it is clear that mathematics may be essentially enlarged and deepened only by enlarging and deepening our notion of number. In this sense all of mathematics after the ancient Greeks grow out of minus 1 and its square root, function theory included. Turning to physics, we now see why quaternions are becoming increasingly important despite their comparative neglect, although they were actually introduced through the back door as the basis of the vector product, the rules of which for 3-space repeat exactly the rules for quaternion multiplication of the unit vectors, except that their squares are zero instead of minus 1. This fact is the basis of Du Val's excellent observa-

[8] The fallacy of Bourbakian set theory is one of reductive omission: distance is ignored, although a distance function, separating the members of any set and allowing them to be distinguished, is implicit in the very notion of *set*; and distance is number, which is thus shown to be the basis of distinguishability and hence of definition. The distance may be governed by a gauge metric or even be stochastic, but it must be there for distinguishability to exist. Even so-called pure projective theorems are special cases of theorems involving angles, which in turn imply separation and distance functions. Moreover, for each so-called "pure" projective theorem there are metric theorems where a certain *non*coincidence occurs, matching a coincidence of the projective theorem, or whereby the angle implied by a given projective ratio is made explicit.

tions in his 1964 book on quaternions and rotations (Oxford University Press) and of our own observations, independently arrived at in 1962; and it is implicit in Hamilton's original work.

It is quite understandable that ordinary physics would find it inconvenient that the square of an operator should become negative. However, in quantum mechanics that is not so inconvenient, and C. W. Kilmister in 1949 was able to demonstrate that Dirac's theory could be made independent of the metric, and hence simpler, by introducing quaternions. In a London conversation with Lancelot Law Whyte, who brought to my attention for the first time his 1954 paper (among others) (*27*), I noted that he too had noticed this fact, and Kilmister and he have the priority for underlining it. In the history of ideas it is usually a new emphasis or implication of past knowledge rather than pure innovation that constitutes historical novelty; for quaternions-as-angles is implied in Hamilton's work.

Certain findings of quantum physics not only substantiate the conclusion of a physical (i.e., not merely pseudo-Euclidean) fourth spatial dimension, but suggest that the spatial dimensionality of our physical universe may well run as high as eight dimensions, for the almost exact rational value of the fine-structure constant 1/137 suggests strongly the existence of the unique eight-dimensional lattice composed of two kinds of cells such that every 137 of them forms an identical group whose constituents are 128 eight-dimensional tetrahedra and 9 eight-dimensional octahedra.

There is no higher finite dimension than 8 that can form a lattice composed solely of tetrahedra and/or octahedra, which are the two simplest regular forms in any dimension, since they have the fewest vertices. Therefore the eighth dimension is an upper limit for lattice regularity and simplicity. Its characteristic number, 137, interestingly points to the fine-structure constant. The exact value of that constant, 1/137.04, suggests that there is a slight curvature of the lattice in at least the ninth dimension, thus allowing slightly more cells per unit of eight-dimensional space. One cell would thus constitute approximately 1/137th of the repeating group pattern of 137 ($=9+128$) cells.

The author has also noted (*17*, p. 242) that the fine structure constant governs the ratio of an electron's mean radius r to the mean radius a of its orbit by the simple equation $(r/a)^{1/2} \simeq 1/137$, thus suggesting that the relation of an electron to the whole configuration of its orbit about the proton is in some sense isomorphic to that of one cell of the eight-dimensional lattice considered as a unit of the entire group pattern of 137 cells. The necessity for at least a fourth physical dimension (specifically for a four-dimensional cylinder, whose cross section is a spheroid) to explain the observed phenomenon of a gravitational field (in press, National Research Council of Italy, Rome) is, however, quite independent of the existence of the spatially eight-dimensional lattice thus indicated by quantum physics.

Thus quaternions do not go far enough. I have long felt that the problems of bio-, psycho-, and sociomorphogenesis will not be solvable until placed on a firm mathematical basis, and that that basis lay in the direction of conditionally randomized hypercomplex variables and their functions, involving a hypercomplex algebra of at least 15 i-elements, which with i_0 comprise the first complete algebra where multiplication becomes non-distributive, and where pairs of annihilation operators can arise.

In a valuable technical paper[9] just called to our attention byARTORGA's knowledgeable editor, Dr. Marcus C. Goodall of the Department of Physics, University of Boston, has already used an algebra—which he calls $Q_8(z)$—isomorphic to Cayley algebra, to resolve some basic problems of quantum field theory in both concrete and elegant fashion. Goodall is also one of the few who is aware of the pertinence of algebraic field theory to quantum mechanics. It may be noted here that this pertinence was implicit ever since the theory of Riemann surfaces was linked with that of algebraic fields by means of multiperiodic functions.

In this connection our proof of the existence of half-integer genus (presented for us at a 1965 meeting of the American Mathematical Society) is relevant, as that concept contains the key to the development of an adequate and more sophisticated theory of transformation groups and automorphisms, including anti-, enantiomorphic, or mirror transformations, and those more complicated ones which are lens-like rather than merely mirror-like. We shall end by observing that hypercomplex number theory and its related algebraic structures will be found increasingly necessary and relevant not only to quantum physics, but to biology, psy-

[9] Not yet published. Based in part on a 1965 report (AFCRL-65-503), U.S. Air Force.

chology, sociology, and even to that dim vista of a scientific theory of history, i.e., *eventology*. There is much to be done and worked out, and the prospects are exciting.

IX. Relations to Function Theory

Functional analysis rests upon function theory, which in turn rests upon algebraic theory. As we have shown, an algebra is no more comprehensive than the nature of the numbers that give rise to it. Complex numbers give rise to a more comprehensive theory of functions than do real numbers, and also generate an algebra (G) which is more comprehensive than ordinary algebra (R), i.e., which embeds *R*.

Thus the theory of numbers in its deepest sense, as the theory of the kinds of possible numbers and their operations (i.e., the algebras pertaining thereto), is the basis of function theory and functional analysis. Change the kind of number and you change the algebra and hence the function theory. Such changes, moreover, are in conformity with the theory of hypercomplex numbers and their appropriate algebraic structures.

Thus number theory as here defined controls any theory of functions and functional analysis. We have already pointed out[8] that Bourbakian set theory is inadequate for functional analysis. Aside from being poorly motivated, pedantically cumbersome, inelegant, and rather artificially ugly, with far more manner than matter, it commits the reductive fallacy of attempting to deny the necessary existence of distance in any valid theory of ensembles of more than one nonnull member or element. The very fact of more than one such element presupposes distinguishability, which in turn implies, at any given moment of the existence of such an ensemble, distance or number in some context. Thus number theory as we have defined it is the *sine qua non* of the theory of ensembles. The natural extension of function theory, and hence of functional analysis, lies then in the direction of the theory of functions of one or more hypercomplex variables. Thus a C-variable is a variable all the possible values of which are Cayley numbers. There would be two kinds of right and left inverse functions of such a variable because the algebra is not simply anticommutative but also anti-associative. Similarly an H-variable would enter into quaternion or H-functions, and these would be able to have no more than one left- and right-hand inverse. Moreover, since the nonassociativity of C-algebra is not mandatory, the extra inverses of C-functions would not exist in certain cases. As an example of H-functions, let us consider the series development of one of the simplest types of right-handed analytic quaternion functions. The coefficients, it will be noted, involve $(1/2\pi^2)$ or the reciprocal unit-sphere surface in D_4, that surface thus being the measure for the integral. These coefficients in the neighborhood of the origin are given by

$$\frac{1}{2\pi^2}\int_{H_q}(q)$$

and by

$$\frac{1}{2\pi^2}\int_{H_p}(p),$$

where H_p and H_q are limiting hypersurfaces including the origin and (p) and (q) are functions involving Fueter's p- and q-functions, which may be taken as the analogues of z^n and z^{-n} in the suitably generalized Laurent expansion, respectively.

The coefficients of C-functions would, as said before, involve in general two distinct varieties of right-left parity. They would also contain as a factor the reciprocal surface of the unit hypersphere in D_8, i.e., $(3/\pi^4)$, the D_8 unit hyperspherical surface (i.e., convex D_7) being now the metric measure. Naturally, complex functions, being in G-space, have as their measure in this sense $1/2\pi$ or the reciprocal of the D_2 sphere surface, i.e., the reciprocal of a circular circumference; and this constant abundantly appears in the theory and theorems of a complex variable.

Very little is known as yet about H-functions, and C-functions have not been considered at all to the writer's knowledge. Neither of course have N-functions, since N-algebra, and hence N-numbers, have hitherto been unknown. Since the entire development of algebraic structures beyond N has been seen to rest upon the nature of zero itself, function theory in N- and higher algebras will involve precise knowledge of the laws of zero formation, of the interaction of zeros of different varieties, and of the results, in terms of the parabolic and loxodromic numbers, which lie beyond the entire hypercomplex number field, as already explained. The mathematics appropriate to biology, psychology, and even to physics in its quantum aspect will not be found distinctively to be statistical, but rather number-theoretical, the word "number" being used here in its most profound sense, which includes all the possible kinds of number (see *Addendum*).

In conclusion we must observe that aside from being rather sterile, linear black-box theory is inapplicable to either nature or manmade devices, all of which importantly and fundamentally involves hysteresis, friction, resistance, viscosity, or some other equally inescapable and pervasive form of increase of entropy; and hence involves nonlinear partial differential equations, which are the rule—any apparent exceptions being simply idealizations, that is to say fictions, and often not useful ones.

Consider now the following nonlinear partial differential system:

$$xz\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0,$$

$$yz\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\frac{\partial z}{\partial y}=0.$$

A solution is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

i.e., an ellipsoid. Now if even such a comparatively simple object as an ellipsoid leads to a nonlinear partial differential equation, we can easily grasp the unreality of suggesting that far more complicated forms and phenomena could be adequately handled by linear methods.

There thus remain two open vistas for the development of the theory of functions: (1) a deepening of number theory, and hence of algebra and function theory, in the direction of more inclusive kinds of numbers; and (2) a development of a theory of nonlinear operators and nonlinear differential equations. These two paths need not be unrelated.

Closely related to the latter are the ordinary differential equations with periodic solutions, such as may arise in the solution of the wave equation by means of curvilinear coordinates. In this connection the work of Professors F. M. Arscott and Kathleen M. Erwin on ellipsoidal and paraboloidal wave functions and their differential equations deserves mention.

For some time we have felt that the theory of turbulence, and in particular of turbulent waves, might benefit if a solution of the wave equation could be found in an orthogonal system which we have termed catenoidal coordinates, formed by the two kinds (1 and 2 sheets) of catenoids of revolution plus the family of surfaces orthogonal to both. Such a system would not simply be based on a quadric equation, but upon one of infinite degree which would, we have reason to believe, have direct relevance to a system of turbulent waves. We have not had time to work out the separation of the wave equation by this means, but enough has been said for any with the necessary interest to do so.

The end of our journey is thus a panorama of open vistas, which is not only appropriate to the hypercomplexly multivalued nature of $\sqrt{-1}$ with which we began, but to mathematics itself, which is in so many ways the least dogmatic and most unexpected of all sciences.

In connection with the hypercomplex, multiple values of $\sqrt{-1}$, a defect of the present theory of ideals should be noted, to which earlier passing reference was made. Ideal numbers have been considered to be either roots of complex numbers or inexpressible, which would be inconsistent. To remedy this defect, ideal numbers must be considered as roots of hypercomplex numbers.

Although all ideals of quadratic fields can be expressed as roots of complex numbers, those of cubic and higher fields cannot in general be so expressed unless "complex" be extended to "hypercomplex." The lack of this theorem, that all ideal numbers are expressible as roots of complex or hypercomplex numbers, is the principal source of obstacles in G. Voronoi's otherwise satisfying exposition of cubic number fields, and in modern works on number fields.

Thus we return, in a new and higher sense, to the conception that ideals are lattices, a conception implicit in Kummer and explicit in Poincaré and other contemporaries. It can further be shown that the mesh area of such lattices is constant. Higher ideals would then become higher-dimensional lattices, and we are again at the fundamental conception of a tangent hyper-sphere group and the loops of units and theory of multiplication of higher algebras, which has already been commented upon.

X. Addendum on Group Theory

The valuable result of Hall (*10*), termed "exciting" by Coxeter, that there exist ternary and not only binary operation groups, is, however, but the beginning of an infinite sequence. The operation that is the basis of Hall's ternary groups is $ax+b$, i.e., an operation combining multiplication and addition, either of which alone is but a binary operation. But a quaternary group is formable on the operational basis of ax^2+bx+c, and an n-ary group, on the basis

$$\sum_{k=0}^{n} a_k x^k.$$

Hence the theory of n-ary groups becomes a mathematico-linguistic transformation of the existing theory of polynomials, and thus is full of interesting isomorphisms, relevant also to function theory.

XI. Addendum on Higher Kinds of Number

The Greeks considered suspect and abnormal any number x such that $k \cdot x < 0$ where k was any positive number. Renaissance man, though he had long accepted negative numbers as just as natural as positive numbers, still balked at x where $x^2 = -1$, although he used such numbers to solve some quadratic equations.

It took until the 19th century until man's mind could regard these numbers too as nonpathological, although the designation "imaginary" still clings to them.

In the 20th century, Eduard Study first considered a number x not equal to zero and such that $x^2=0$; although Study still had no realization that this implies also $x^0=0$, and an advanced form of nondistributive multiplication. The present survey has revealed the evolutionary ancestors of p namely, numbers such as a and b, neither zero nor infinite nor equal, and such that $a \cdot b = 0$; the next higher nondistributive number being given by $a \cdot 0 = b$.

The foregoing paper has also developed numbers beyond p, namely, a nonzero q, such that $|q|=1$; $q^4=0$; $q \neq q^2 \neq q^3 \neq 0$; and, unlike all the preceding numbers, with q^2 and q^3 irreducible to any real number or any lower power of q.

It is also noteworthy that $1/u \neq u^{-1}$ is true for $u=q$; and that $1/u^2 \neq u^{-2}$

and $1/u^3 \neq u^{-3}$ are true also for $u=q$. The nonrepresentability of reciprocals in terms of powers of their denominators is deeply related to the enantiomorphic phenomena that begin to be noticeable in what may be called the third metadimension, that of the p-numbers, the realm of the q-numbers constituting the fourth metadimension, whereas that of (hyper)imaginary or i-numbers constitutes the second, while the real axis represents the first metadimension, since it may represent any dimension of real space; all these being copies of each other. But the dimensions of the higher metadimensions are not copies of each other. For even the two first dimensions, i_1 and i_2, of the second metadimension cannot be exact copies, since it is no longer true that $i_1 \cdot i_2$ and $i_2 \cdot i_1$ are equal.

The metadimensions are isomorphic to the kinds of number, which in turn are characterized by their unit power fields, i.e., the function their unit traces out in a suitably defined representation plane, when it is reiteratedly multiplied by itself. Thus the unit field form shows the self-reflexive operation of the given unit or kind of number.

The forms for the five kinds of number with unit power fields of real, finite degree are shown in Table III. In another context, in 1962, we defined the first six metadimensions: those with real field forms.

TABLE III

Metadimension	Kind of number	Characteristic unit operation	Unit field form	Degree
1	Real	$u^2=1$	$x^2=1$ or $x=\pm 1$	Bilinear
2	Imaginary and hyperimaginary	$u^2=-1$	$x^2+y^2=1$	Quadratic
3	p-Numbers (metaimaginary)	$u^2=0$	$(x^2+y^2)^2/y^2=1$	Quartic
4	q-Numbers (metaimaginary)	$u^4=0$	$y^2=x^4-x^6$	Sextic
5	w-Numbers (metaimaginary)	$1/u^8=0$	$y=x^4\pm(x+2)(x^2-1)^{1/2}$	Octic

It will be observed that the q-numbers are the last with a finite, symmetric unit field form. The w-numbers have no longer either a finite or symmetric field form, and hence develop another (asymmetric enantiomorph) type of field form when the factor $(x+2)$ is replaced by $(x-2)$. The *orders* of w-

numbers (analogous to $i_1 \equiv i$, $i_2 \equiv j$, $i_3 \equiv k$, etc.) may be represented by

$$y = x^4 \pm (x+1+n)(x^2-1)^{1/2}, \quad |n| = 1, 2, 3 \ldots .$$

The negative orders from (-2) asymmetrically mirror the nonnegative orders from the zeroth onward. But the (-1)st order yields the equation $y^2 = x^4 \pm x(x^2-1)^{1/2}$ which interestingly yields, on the substitution $x \to ix$ the cognate form $x^4 \pm x(x^2+1)^{1/2}$.

Beyond the w-numbers, more vast changes occur. The unit field form of the sixth metadimension is no longer representable by an equation of finite degree; and the seventh metadimension requires a unit field equation which is a function of a nonreal variable.

Finally, it can be shown that the eighth metadimension is necessarily nonrepresentable in any representation space, and that it contains all metadimensions beyond itself by an inherent, self-induced continuation.

It thus turns out that there are eight possible basic kinds of number (each with their own infinities), plus zero.

The higher kinds of number for the first time yield concrete hope of placing the profound and subtle characteristics of bio-, psycho-, and socio-transformations and processes on an adequate mathematical basis. Such kinds of number would thus introduce the humanities to their appropriate mathematics, which will not do them the grave and unscientific injustice of forcing them to fit some Procrustean bed of inadequate hypothesis or reductive definition. Man and man's sciences are now ready to go beyond the square root of minus one. With each new and higher kind of number a new and deeper algebra and arithmetic become possible, and hence a new and deeper functional analysis.

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