# F4 and E8:
## Commutators and AntiCommutators

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Abstract:

Realistic Physics models must describe both commutator Bosons and anticommutator Fermions so that spin and statistics are consistent.
The usual commutator structure of Lie Algebras can only describe Bosons, so a common objection to Physics models that describe both Bosons and Fermions in terms of a single unifiying Lie Algebra (for example, Garrett Lisi's E8 TOE) is that they violate consistency of spin and statistics by using Lie Algebra commutators to describe Fermions.
However,
Pierre Ramond has shown in hep-th/0112261 as shown that the exceptional Lie Algebra F4 can be described using anticommutators as well as commutators.
This essay uses the periodicity property of Real Clifford Algebras to show that E8 can also be described using anticommutators as well as commutators so that it may be possible to construct a realistic Physics model that uses the exceptional Lie Algebra E8 to describe both Bosons and Fermions.

E8 also inherits from F4 Triality-based symmetries between Bosons and Fermions that can give the useful results of SuperSymmetry without requiring conventional SuperPartner particles that are unobserved by LHC.

Realistic Physics models must describe both
integer-spin Bosons whose statistics are described by commutators
(examples are Photons, W and Z bosons, Gluons, Gravitons, Higgs bosons)
and
half-integer-spin Fermions whose statistics are described by anticommutators.
(examples are 3 generations of Electrons, Neutrinos, Quarks and their antiparticles)

Lie Algebra elements are usually described by commutators of their elements
so
if a Physics model attempts to describe both Bosons and Fermions as elements of a single unifiying Lie Algebra (for example, Garrett Lisi's E8 TOE)
a common objection is:

since the Lie Algebra is described by commutators,
it can only describe Bosons and cannot describe Fermions
therefore
models (such as Garrett Lisi's) using E8 as a single unifying Lie Algebra
violate the consistency of spin and statistics and are wrong.

However, Pierre Ramond has shown in hep-th/0112261 as shown that the exceptional Lie Algebra F4 can be described using anticommutators as well as commutators.

The periodicity property of Real Clifford Algebras shows that E8 inherits from F4 a description using anticommutators as well as commutators so that
it may be possible to construct a realistic Physics model that uses
the exceptional Lie Algebra E8 to describe both Bosons and Fermions.

Here are relevant quotes from hep-th/0112261 by Pierre Ramond:
"... exceptional algebras relate tensor and spinor representations
of their orthogonal subgroups,
while Spin-Statistics requires them to be treated differently ...
all representations of the exceptional group F4 are generated by three sets
of oscillators transforming as 26. We label each copy of 26 oscillators as

$$A_0^{[\kappa]},\ A_i^{[\kappa]},\ i=1,\cdots,9,\ B_a^{[\kappa]},\ a=1,\cdots,16,$$

and their hermitian conjugates, and where k = 1, 2, 3.
Under SO(9), the A[k]_i transform as 9, B[k]_a transform as 16, and A[k]_0 is a scalar. They satisfy the commutation relations of ordinary harmonic oscillators

$$[\,A_i^{[\kappa]}\,,\,A_j^{[\kappa']\,\dagger}\,]\;=\;\delta_{ij}\,\delta^{[\kappa]\,[\kappa']}\;,\qquad[\,A_0^{[\kappa]}\,,\,A_0^{[\kappa']\,\dagger}\,]\;=\;\delta^{[\kappa\,\kappa']}\;.$$

Note that the SO(9) spinor operators satisfy Bose-like commutation relations

$$[\,B_a^{[\kappa]}\,,\,B_b^{[\kappa']\,\dagger}\,]\;=\;\delta_{ab}\,\delta^{[\kappa]\,[\kappa']}\;.$$

The generators T_ij and T_a

$$T_{ij}\;=\;-i\sum_{\kappa=1}^{4}\left\{\left(A_i^{[\kappa]\dagger}A_j^{[\kappa]}-A_j^{[\kappa]\dagger]}A_i^{[\kappa]}\right)+\frac{1}{2}\,B^{[\kappa]\dagger}\,\gamma_{ij}\,B^{[\kappa]}\right\}\;,$$

$$T_a\;=\;-\frac{i}{2}\sum_{\kappa=1}^{4}\left\{(\gamma_i)^{ab}\left(A_i^{[\kappa]\dagger}B_b^{[\kappa]}-B_b^{[\kappa]\dagger}A_i^{[\kappa]}\right)'-\sqrt{3}\left(B_a^{[\kappa]\dagger}A_0^{[\kappa]}-A_0^{[\kappa]\dagger}B_a^{[\kappa]}\right)\right\}\;,$$

satisfy the F4 algebra,

$$\begin{aligned}[T_{ij}\,,T_{kl}]\;&=\;-i\left(\delta_{jk}\,T_{il}+\delta_{il}\,T_{jk}-\delta_{ik}\,T_{jl}-\delta_{jl}\,T_{ik}\right)\;,\\ [T_{ij}\,,T_a]\;&=\;\frac{i}{2}\,(\gamma_{ij})_{ab}\,T_b\;,\\ [T_a\,,T_b]\;&=\;\frac{i}{2}\,(\gamma_{ij})_{ab}\,T_{ij}\;,\end{aligned}$$

so that the structure constants are given by

$$f_{ij\,ab}\;=\;f_{ab\,ij}\;=\;\frac{1}{2}\,(\gamma_{ij})_{ab}\;.$$

The last commutator requires the Fierz-derived identity

$$f_{ij\,ab}\;=\;f_{ab\,ij}\;=\;\frac{1}{2}\,(\gamma_{ij})_{ab}\;.$$

from which we deduce

$$3\,\delta^{ac}\delta^{db}+(\gamma^i)^{ac}\,(\gamma^i)^{db}-(a\leftrightarrow b)\;=\;\frac{1}{4}\,(\gamma^{ij})^{ab}\,(\gamma^{ij})^{cd}\;.$$

To satisfy these commutation relations, we have required both A_0 and B_a
to obey Bose commutation relations

**(Curiously,**

**if both [ A_0 and B_a ] are anticommuting, the F4 algebra is still satisfied).** ...".

The 1 + 9 + 16 = 26 oscillators

$$A_0^{[\kappa]},\ A_i^{[\kappa]},\ i=1,\cdots,9,\ B_a^{[\kappa]},\ a=1,\cdots,16,$$

represent the 26-dim lowest-dimensional non-trivial represenstion of 52-dim F4 .

The 36 + 16 = 52 generators

$$T_{ij} = -i\sum_{\kappa=1}^{4}\left\{\left(A_i^{[\kappa]\dagger}A_j^{[\kappa]} - A_j^{[\kappa]\dagger]}A_i^{[\kappa]}\right) + \frac{1}{2}B^{[\kappa]\dagger}\gamma_{ij}B^{[\kappa]}\right\},$$

$$T_a = -\frac{i}{2}\sum_{\kappa=1}^{4}\left\{(\gamma_i)^{ab}\left(A_i^{[\kappa]\dagger}B_b^{[\kappa]} - B_b^{[\kappa]\dagger}A_i^{[\kappa]}\right)' - \sqrt{3}\left(B_a^{[\kappa]\dagger}A_0^{[\kappa]} - A_0^{[\kappa]\dagger}B_a^{[\kappa]}\right)\right\},$$

represent the 52-dim adjoint representation of F4 written as commutators.

By the remark shown in bold in the quote above, Ramond states that
the 16 Spinor oscillators B[k]_a can be written as anticommutators
as well as commutators and that both cases produce the 52-dim F4 algebra.

Physically, this means that if you use the 52-dim F4 to build a physics model
with Fermions being represented by 16-dim F4 / SO(9) = OP2
then
you can use the anticommutator structure of the 16-dim B[k]_a
to satisfy the spin-statistics theorem
because
the B[k]_a represent a 16 of SO(9) which is also OP2 = F4 / SO(9) .

To see how this anticommutator structure extends to E8,
note that the lowest-dimensional non-trivial representation of E8 is 248-dim
and that the adjoint representation of E8 is also 248-dim.

As shown by T. Fulton in J. Phys. A: Math. Gen. 18 (1985) 2863-2891
(quotation slightly modified due to typographical considerations etc):

"... Schwinger ... has studied the generators and irreps of SU(2) in terms of two Bose oscillators (hereafter abbreviated SHO) ... the algebras of the classical groups ... A(n) ; B(n) ; D(n) ; C(n) have been realized in terms of Fermi oscillators. The spinor irreps of the orthogonal groups ... are the only ones which ... have been constructed ... using Fermi oscillators. These irreps involve various nunbers of Fermi oscillator creation operators a†_i
...
the elementary spinor irreps of the orthogonal groups can be written in terms of a single SHO creation operator acting on the vacuum state ... the 'vacuum state' does not have zero weight, but is an element of a spinor irrep.
...
For D(n), the 'vacuum state', together with all states formed by even powers of a† operating on this state, up to the maximum possible such power, constitute the elements of one of the elementary spinor irreps; all possible odd powers of a† , operating on the 'vacuum', constitute the set of all elements of the other elementary spinor irrep ...
...
F(4) ... we have elementary irrep f1 = 26-dim
so that D(13) = SO(26) contains F4
The other elementary irrep of F(4) is f2 = 52-dim
...
E(8) ... we have elementary irrep f1 = 248-dim
The simplest embedding of E(8) is to choose D(124) = SO(248) contains E(8) ...".

Physically, if you use the 248-dim E8 to build a physics model
with Fermions being represented by 128-dim E8 / D8 = (OxO)P2
and
if E8 inherits anticommutator structure from F4
then
you can use anticommutator structure of the 128-dim (OxO)P2
to satisfy the spin-statistics theorem.

To see how the anticommuting property of the 16 B_a elements of F4
can be inherited by some of the elements of E8,
consider that 52-dimensional F4 is made up of:

28-dimensional D4 Lie Algebra Spin(8) (in commutator part of F4)
8-dimensional D4 Vector Representation V8 (in commutator part of F4)
8-dimensional D4 +half-Spinor Representation S+8 (in anticommutator part of F4)
8-dimensional D4 -half-Spinor Representation S-8 (in anticommutator part of F4)

Since 28-dimensional D4 Spin(8) is the BiVector part BV28
of the Real Clifford Algebra Cl(8) with graded structure
Cl(8) = 1 + V8 + BV28 + 56 + 70 + 56 + 28 + 8 + 1
and with Spinor structure
Cl(8) = (S+8 + S-8) x (8 + 8)

F4 can be embedded in Cl(8) (blue commutator part, red anticommutator part):

F4 = V8 + BV28 + S+8 + S-8

Note that V8 and S+8 and S-8 are related by the Triality Automorphism.

Also consider the 8-periodicity of Real Clifford Algebras,
according to which for all N

Cl(8N) = Cl(8) x ...(N times tensor product)... Cl(8)

so that in particular Cl(16) = Cl(8) x Cl(8)
where Cl(16) graded structure is 1 + 16 + BV120 + 560 + ... + 16 + 1
and Cl(16) Spinor structure is ( (S+64 + S-64) + (64 + 64) ) x (128 + 128)
and Cl(16) contains 248-dimensional E8 as

E8 = BV120 + S+64 + S-64

where BV120 = 120-dimensional D8 Lie Algebra Spin(16)
and S+64 + S-64 = 128-dimensional D8 half-Spinor Representation

Consider two copies of F4 embedded into two copies of Cl(8).

**For commutator structure:**

The tensor product of the two copies of Cl(8) can be seen as

1 + V8 + BV28 + 56 + 70 + 56 + 28 + 8 + 1
x
1 + V8 + BV28 + 56 + 70 + 56 + 28 + 8 + 1

which produces the Real Clifford Algebra Cl(16) with graded structure

1 + 16 + BV120 + 560 + 1820 + ... + 16 + 1

where the Cl(16) BiVector BV120 is made up of 3 parts

BV120 = BV28x1 + 1xBV28 + V8xV8

that come from the V8 and BV28 commutator parts of the two copies of F4.

This gives the commutator part of E8 as BV120 inheriting commutator structure from the two copies of F4 embedded in two copies of Cl(8) whose tensor product produces Cl(16) containing E8.

**For anticommutator structure:**

The tensor product of the two copies of 256-dim Cl(8) can also be seen as

( (S+8 + S-8) x (8 + 8) )
x
( (S+8 + S-8) x (8 + 8) )

which produces the 2^16 = 65,536 = 256x256-dim Real Clifford Algebra Cl(16)

( (S+8 + S-8) x (S+8 + S-8))
x
( (8 + 8) x (8 + 8) )

with 256-dimensional Spinor structure

( (S+8 + S-8) x (S+8 + S-8)) =

= ( (S+8 x S+8) + (S-8 x S-8) ) + ( (S+8 x S-8) + (S-8 x S+8) )

that comes from the S+8 and S-8 anticommutator parts of the two copies of F4.

Since the (S+8 x S-8) and (S-8 x S+8) terms inherit mixed helicities from F4

only the (S+8 x S+8) and (S-8 x S-8) terms inherit consistent helicity from F4.

Therefore, define S+64 = (S+8 x S+8) and S-64 = (S-8 x S-8)
so that
( S+64 + S-64 ) = 128-dimensional D8 half-Spinor Representation

This gives the anticommutator part of E8 as S+64 + S-64 inheriting anticommutator structure from the two copies of F4 embedded in two copies of Cl(8) whose tensor product produces Cl(16) containing E8.

The result is that 248-dimensional E8 is made up of:

BV120 = 120-dimensional D8 Lie Algebra Spin(16) (commutator part of E8)

128-dimensional ( S+64 + S-64 ) D8 half-Spinor (anticommutator part of E8)

Note that since the V8 and S+8 and S-8 components of F4 are related by Triality,
and since
the E8 component BV120 contains 64-dimensional V8xV8
and
the 64-dimensional E8 component S+64 = S+8 x S+8
and
the 64-dimensional E8 component S-64 = S-8 x S-8

E8 inherits from the two copies of F4 a Triality relation

V8xV8 = S+64 = S-64

The commutator - anticommutator structure of E8 allows construction of realistic Physics models that not only unify both Bosons and Fermions within E8
but
also contain Triality-based symmetries between Bosons and Fermions
that can give the useful results of SuperSymmetry
without requiring conventional SuperPartner particles that are unobserved by LHC.

**CONCLUSION:**

Unified E8 Physics models can be constructed without violating spin-statistics,
so that evaluation of such models as Garrett Lisi's E8 TOE
and my E8 Physics model at http://vixra.org/abs/1108.0027 should be based
on other criteria such as consistency with experimental observations.

Cl(16) contains E8

Cl(16) = 1 + 16 + **120** + 560 + 1820 + ... + 16 + 1=

= ( (**64++ + 64--**) + (64+- + 64-+) ) x 256

Cl(16) = Cl(8)1 x Cl(8)2 where Cl(8)1 contains F41 and Cl(8)2 contains F42

( 8ofF41 x 8ofF42 ) + 28ofF41x1ofCl(8)2 + 28ofF42x1ofCl(8)1 = 64 + 28 + 28 = 120ofE8

( **8+**ofF41 + **8-**ofF41 ) x ( **8+**ofF42 + **8-**ofF42 ) gives ( **64++ + 64--** )ofE8

(because the terms ( 64+- + 64-+ ) are rejected as unphysical mixed helicity)

( The notation -> denotes a Maximal Contraction )

E8 = **120** + ( **64++ + 64--** )

E8 -> A7 = SU(8) part of U(8) for 8 position x 8 momentum = 64 generators

- semidirect product with -

28 + 64 + 1 + 64 + 28 Heisenberg Algebra H92 for

28 gauge bosons for A2xA1xA0 = SU(3)xSU(2)xU(1) plus

D3 = Spin(2,4) = A3 = SU(2,2) Conformal Gravity

and for

64 = 8x8 of 8 components of 8 Fermions

E7.5 ->E7 - semidirect product with - 28+1+28 Heisenberg Algebra H28

28 = Quaternionic Jordan Algebra J4(Q) =(bijection)= D4

D4 = Spin(8) = (Standard Model + Gravity) Gauge Bosons

E7 -> E6 - semidirect product with - 27+1+27 Heisenberg Algebra H27

27 has 3 diagonal generators of Quaternion SU(2) + ( 8 + 8 ) Fermions) + 8 Spacetime vectors

27 = Octonionic Jordan Algebra J3(O) =(bijection)= J4(Q)o

E6 -> D5 = Spin(10) - semidirect product with - 16+1+16 Heisenberg Algebra H16

16 = (26-10)-dim Fermion part of J3(O)o

E6 = F4 + J3(O)o 26-dim traceless part of 27-dim Jordan Algebra J3(O)

F4 = **8** + **28** + ( **8+** + **8-** )

F4 -> B3 = ( Spin(6) + S6 ) - semidirect product with - 15+1+15 Heisenberg Algebra H15

for 15 D3 = Spin(2,4) Conformal Gauge Bosons

6-dim Conformal Spacetime reduces to M4 Physical Spacetime

Cl(8) = 1 + **8** + **28** + 56 + 70 + 56 + 28 + 8 + 1 = ( **8+** + **8-** ) x 16