Cl(Cl(4)) = Cl(16) containing E8

Frank Dodd (Tony) Smith, Jr. - 2011

Cl(4):
1 grade-0: s
4 grade-1: x y z t - M4 physical spacetime
6 grade-2: a b c d e f - M4L Lorentz transformations
4 grade-3: x y z t - CP2 internal symmetry space
1 grade-4: s

Cl(Cl(4)) = Cl(16) for which Physical Interpretations are based on Triality whereby
x y z t x y z t corresponds to
8-dim M4xCP2 Kaluza-Klein SpaceTime
8 elementary Fermion Particles
8 elementary Fermion AntiParticles.
The 8-dim M4xCP2 Kaluza-Klein interpretation is used for Cl(16) grade-1 in which
x y z t x y z t occur as single elements
The 8 Fermion Particle - 8 Fermion AntiParticle interpretation is used for the gauge forces of grade-2 in which x y z t x y z t occur as antisymmetric pairs.

1 grade-0:
s

16 grade-1:
s
x y z t - M4 physical spacetime
a b c d e f
x y z t - CP2 internal symmetry space
s

Further Physical Interpretations:
Even-Odd Clifford Dual to M4 physical spacetime:
s a b c
Even-Odd Clifford Dual to CP2 internal symmetry space:
d e f s

120 grade-2:

sx sy sz st
sa sb sc sd se sf
sx sy sz st   ss

xy xz xt
xa xb xc xd xe xf
xx xy xz xt   xs

yz yt
ya yb yc yx ye yf
yx yy yz yt   ys

zt
za zb zc zd ze zf
zx zy zz zt   zs

ta tb tc td te tf
tx ty tz tt   ts

ab ac ad ae af
ax ay az at   as

bc bd be bf
bx by bz bt   bs

cd ce cf
cx cy cz ct   cs

de df
dx dy dz dt   ds
ef
ex ey ez et   es
fx fy fz ft   fs
xy xz xt   xs
yz yt   ys
zt   zs
                           ts

Physical Interpretations of the 120 grade-2 elements:

28-dim D4 Spin(8) for Standard Model Gauge Groups:

```
xy xz xt
yz yt
zt

xx xy xz xt |
yx yy yz yt | - This is U(4) that contains SU(3).
zx zy zz zt |   U(2) = SU(2)xU(1) arises from
tx ty tz tt |   CP2 = SU(3)/U(2) by Batakis.

xy xz xt
yz yt
zt
```

28-dim D4 Spin(8) for Conformal Gravity:

```
sa sb sc sd se sf

ss

ab ac ad ae af |
bc bd be bf    | - This is Spin(2,4) Conformal Group
cd ce cf       |   that gives
de df          |   Gravity by MacDowell-Mansouri.
ef             |

as
bs
cs
ds
es
fs
```

64-dim to describe 8-dim Kaluza-Klein SpaceTime:
Consider 8-dim K-K as Octonion Spacetime
with Octonion basis {1,i,j,k,E,I,J,K}.
For each of the 8 x y z t x y z t Position dimensions
there are 8 Momentum dimensions represented by
s a b c s d e f and basis elements {1,i,j,k,E,I,J,K}.
The a b c correspond to an SU(2) and so to {i,j,k}.
The d e f correspond to another SU(2) and to (I,J,K}.

8 s-terms for Real Part of Octonion SpaceTime:
sx sy sz st
sx sy sz st

8 s-terms for E-Imaginary Part of Octonion SpaceTime:
xs
ys
zs
ts
xs
ys
zs
ts

24 M4 ijkIJK components of Octonion SpaceTime:
xa xb xc xd xe xf
ya yb yc yx ye yf
za zb zc zd ze zf
ta tb tc td te tf

24 CP2 ijkIJK components of Octonion SpaceTime:
ax ay az at
bx by bz bt
cx cy cz ct
dx dy dz dt
ex ey ez et
fx fy fz ft

E8 is constructed from Cl(16) using grade-2 and half-Spinors
so consider Spinors of Real Clifford Algebras:

$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{C})$	$M_{16}(\mathbf{H})$	$M_{16}(\mathbf{H}) \oplus M_{16}(\mathbf{H})$	$M_{32}(\mathbf{H})$	$M_{64}(\mathbf{C})$	$M_{128}(\mathbf{R})$	$M_{128}(\mathbf{R}) \oplus M_{128}(\mathbf{R})$	$M_{256}(\mathbf{R})$
$M_{8}(\mathbf{C})$	$M_{8}(\mathbf{H})$	$M_{8}(\mathbf{H}) \oplus M_{8}(\mathbf{H})$	$M_{16}(\mathbf{H})$	$M_{32}(\mathbf{C})$	$M_{64}(\mathbf{R})$	$M_{64}(\mathbf{R}) \oplus M_{64}(\mathbf{R})$	$M_{128}(\mathbf{R})$	$M_{128}(\mathbf{C})$
$M_{4}(\mathbf{H})$	$M_{4}(\mathbf{H}) \oplus M_{4}(\mathbf{H})$	$M_{8}(\mathbf{H})$	$M_{16}(\mathbf{C})$	$M_{32}(\mathbf{R})$	$M_{32}(\mathbf{R}) \oplus M_{32}(\mathbf{R})$	$M_{64}(\mathbf{R})$	$M_{64}(\mathbf{C})$	$M_{64}(\mathbf{H})$
$M_{2}(\mathbf{H}) \oplus M_{2}(\mathbf{H})$	$M_{4}(\mathbf{H})$	$M_{8}(\mathbf{C})$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{R}) \oplus M_{16}(\mathbf{R})$	$M_{32}(\mathbf{R})$	$M_{32}(\mathbf{C})$	$M_{32}(\mathbf{H})$	$M_{32}(\mathbf{H}) \oplus M_{32}(\mathbf{H})$
$M_{2}(\mathbf{H})$	$M_{4}(\mathbf{C})$	$M_{8}(\mathbf{R})$	$M_{8}(\mathbf{R}) \oplus M_{8}(\mathbf{R})$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{C})$	$M_{16}(\mathbf{H})$	$M_{16}(\mathbf{H}) \oplus M_{16}(\mathbf{H})$	$M_{32}(\mathbf{H})$
$M_{2}(\mathbf{C})$	$M_{4}(\mathbf{R})$	$M_{4}(\mathbf{R}) \oplus M_{4}(\mathbf{R})$	$M_{8}(\mathbf{R})$	$M_{8}(\mathbf{C})$	$M_{8}(\mathbf{H})$	$M_{8}(\mathbf{H}) \oplus M_{8}(\mathbf{H})$	$M_{16}(\mathbf{H})$	$M_{32}(\mathbf{C})$
$M_{2}(\mathbf{R})$	$M_{2}(\mathbf{R}) \oplus M_{2}(\mathbf{R})$	$M_{4}(\mathbf{R})$	$M_{4}(\mathbf{C})$	$M_{4}(\mathbf{H})$	$M_{4}(\mathbf{H}) \oplus M_{4}(\mathbf{H})$	$M_{8}(\mathbf{H})$	$M_{16}(\mathbf{C})$	$M_{32}(\mathbf{R})$
$\mathbf{R} \oplus \mathbf{R}$	$M_{2}(\mathbf{R})$	$M_{2}(\mathbf{C})$	$M_{2}(\mathbf{H})$	$M_{2}(\mathbf{H}) \oplus M_{2}(\mathbf{H})$	$M_{4}(\mathbf{H})$	$M_{8}(\mathbf{C})$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{R}) \oplus M_{16}(\mathbf{R})$
$\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$	$\mathbf{H} \oplus \mathbf{H}$	$M_{2}(\mathbf{H})$	$M_{4}(\mathbf{C})$	$M_{8}(\mathbf{R})$	$M_{8}(\mathbf{R}) \oplus M_{8}(\mathbf{R})$	$M_{16}(\mathbf{R})$

Real Spinors (signatures (2,2) (3,1))
Cl(4) = M4(R) = 4x4 Real Matrix Algebra
Cl(8) = M16(R) (signature (0,8))
Cl(16) = M16(R) (x) M16(R) = M256(R) (signature (0,16))

Physically, the Real Structures describe
High-Energy (near Planck scale) Octonionic Physics.

Cl(4) Spinors:
4-dim x y z t space on which M4(R) matrices act.
With Spinors defined in terms
of Even Subalgebra of Clifford Algebra,
M4(R) reduces to M2(R) + M2(R)
and Cl(4) Spinors reduce to sum of half-Spinors as
2-dim x y space plus 2-dim z t space.

Cl(8) Spinors:
16-dim space on which M16(R) matrices act.
M16(R) reduces to M8(R) + M8(R)
and Cl(8) Spinors reduce to sum of half-Spinors as
8-dim x y z t x y z t +space plus
8-dim x y z t x y z t -space
where Triality has been used to represent half-Spinors
in terms of vectors x y z t x y z t that can be seen
as Cl(4) structures.

Cl(Cl(4)) = Cl(16) Spinors:
256-dim space on which M256(R) matrices act.
M256(R) reduces to M128(R) + M128(R)
and Cl(16) Spinors (8+ + 8-)x(8+ + 8-) =
= (64++ + 64--) + (64+- + 64-+) = 128pure + 128mixed
which reduces to sum of half-Spinors as
128-dim pure space plus 128-dim mixed space.
Only the pure half-Spinor 128-dim space is used to
construct E8 = 120-dim grade-2 + 128-dim half-Spinor.
The pure 128-dim half-Spinor 64++ + 64-- describes:
8 covariant components of 8 Fermion Particles by 64++
8 covariant components of 8 AntiParticles by 64-- .

Quaternion Spinors (signatures (0,4) (1,3) (4,0))

Cl(4) = M2(H) = 2x2 Quaternion Matrix Algebra

Cl(8) = M8(H) (signature (2,6))

Cl(16) = M8(H) (x) M8(H) = M128(H) (signature (4,12))

Physically, Quaternionic Structures describe
Low-Energy (with respect to Planck scale) Physics
which emerges after
Octonion Symmetry is broken
by "freezing out" a preferred Quaternion Substructure
at the End of Inflation
so
Quaternionic Structure is relevant for Low-Energy physics described by Cl(4) and observed directly by us now,
but not relevant for Cl(8) or Cl(16) which describe High-Energy physics such as that of the Inflationary Era.

Cl(4) Spinors:
8-dim space on which M2(H) matrices act.
With Spinors defined in terms
of Even Subalgebra of Clifford Algebra,
M2(H) reduces to H+H
and Cl(4) Spinors reduce to sum of half-Spinors as
4-dim space plus 4-dim space
which enables Cl(4) to describe Fermion Particles as
Lepton + RGB Quarks Particles by one H of H+H plus
Lepton + RGB Quarks AntiParticles by the other H of H+H
but Cl(4) is not large enough to distingush Neutrinos
from Electrons. To do that it should be expanded into
Cl(6) of the Conformal Group (signature (2,4))
with Cl(6) = M4(H) and Even Subalgebra M2(H) + M2(H)
giving a half-Spinor H+H for 8 Fermion Particles and
another half-Spinor H+H for 8 Fermion AntiParticles.
In a sense, this expands 4+4=8-dim Batakis Kaluza-Klein
to a 6+4=10-dim CNF6 x CP2 Kaluza-Klein,
with the M4 Minkowski M4 physical SpaceTime becoming a
conformal CNF6 physical SpaceTime
that is related to Segal Conformal Dark Energy.

Higher grades of Cl(16) are:

560 grade-3:

1820 grade-4:

4368 grade-5:

8008 grade-6:

11440 grade-7:

12870 grade-8:

11440 grade-9:

8008 grade-10:

4368 grade-11:

1820 grade-12:

560 grade-13:

120 grade-14:

16 grade-15:

1 grade-16:

Higgs as Primitive Idempotent:

Clifford Algebra Primitive Idempotents are described by Pertti Lounesto in his book Clifford Algebras and Spinors (Second Edition, LMS 286, Cambridge 2001) in whch he said at pages 226-227 and 29:
"... Primitive idempotents and minimal left ideals An orthonormal basis of R(p,q) induces a basis of Cl(p,q), called the standard basis.
Take a non-scalar element e_T, e_T^2 = 1, from the standard basis of Cl(p,q).
Set e = (1/2)(1 + e_T) and f = (1/2)(1 - e_T), then e + f = 1 and ef = fe = 0.
So Cl(p,q) decomposes into a sum of two left ideals
Cl(p,q) =Cl(p,q) e + Cl(p,q) f , where [for n = p + q]
dim Cl(p,q) e = dim Cl(p,q) f = [dim] (1/2) Cl(p,q) = 2^(n-1).
Furthermore,
if { e_T_1 , e_T_2 , ... , e_T_k } is a set of non-scalar basis elements
such that e_T_i^2 = 1 and e_T_i e_T_j = e_T_j e_T_i ,
then letting the signs vary independently in the product
(1/2)(1 +/- e_T _1) (1/2)(1 +/- e_T_2) ... (1/2)(1 +/- e_T_k) ,
one obtains 2^k idempotents which are mutually annihilating and sum up to 1.
The Clifford algebra Cl(p,q) is thus decomposed into a direct sum of 2^k left ideals, and by construction, each left ideal has dimension 2^(n - k) .
In this way one obtains a minimal left ideal by forming a maximal product of non-annilating and commuting idempotents.
The Radon-Hurwitz number r_i for i in Z is given by

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

and the recursion formula r_(i + 8) = r_ i + 4 .
For the negative values of i one may observe that r_(-1) = -1
and r_(-i) = 1 - i + r_(i + 2) for i > 1 .
r_-8 = 1 - 8 + r_10

Theorem. In the standard basis of Cl(p,q) there are always
k = q - r_(q - p) non-scalar elements e_T_i , e_T_i^2 = 1 ,
which commute, e_T_i e_T_j = e_T_j e_T_i ,
and generate a group of order 2^k .

The product of the corresponding mutually non-annihilating idempotents,
f = (1/2)(1 +/- e_T _1) (1/2)(1 +/- e_T_2) ... (1/2)(1 +/- e_T_k) ,

is primitive in Cl(p,q).

Thus, the left ideal S = Cl(p,q) f is minimal in Cl(p,q).
Example ... In the case of R(0,7) we have k = 7 - r_7 = 4. Therefore the idempotent
f = (1/2)(1 + e_124) (1/2)(1 + e_235) (1/2)(1 + e_346) (1/2)(1 + e_457)
is primitive to Cl(0,7) = 2^Mat(8,R). ...”.

Further example of R(0,8) is discussed by Pertti Lounesto in his book
“Spinor Valued Regular Functions in Hypercomplex Analysis”
(Report-HTKKMAT-A154 (1979) Helsinki University of Technology) said
[in the quote below I have changed his notation for a Clifford algebra
from R_(p,q) to Cl(p,q)] at pages 40-42:
"... To fix a minimal left ideal V of Cl(p,q)
we can choose a primitive idempotent f of Cl(p,q) so that V = Cl(p,q) f .
By means of an orthonormal basis { e_1 , e_2 , ... , e_n }
for [the grade-1 vector part of Cl(p,q)] Cl^1(p,q) we can construct
a primitive idempotent f as follows:
Recall that the 2^n elements
e_A = e_a_1 e_a_2 ... e_a_k ,
1 < a_1 < a_2 < ... < s_k < n
constitute a basis for Cl(p,q). ...
dim_R V = 2^X , where X = h or X = h + 1 according as
p - q = 0, 1, 2 mod 8 or p - q = 3, 4, 5, 6, 7 mod 8 and h = [n / 2] .
Select n - X elements e_A, e_A^2 = 1 , so they are pairwise commuting
and generate a group of order 2^(n - X) .
Then the idempotent ...
f = (1/2)(1 + e_A_1) (1/2)(1 + e_A_2) ... (1/2)(1 + e_A_(n - X))
is primitive ...
To prove this note that the dimension of (1/2)(1 + e_A) Cl(p,q) is (2^n) / 2
and so the dimension of Cl(p,q) f is (2^n) / (2^(n - X)) = 2^X .
Hence,
if there exists such an idempotent f , then f is primitive.
To prove that such an idempotent f exists in every Clifford algebra Cl(p,q)
we may first check the lower dimensional cases and then proceed by making use
of the isomorphism Cl(p,q) x Cl(0,8) = Cl(p, q + 8)
and the fact that Cl(0,8) has a primitive idempotent
f = (1/2)(1 + e_1248) (1/2)(1 + e_2358) (1/2)(1 + e_3468) (1/2)(1 + e_4578)
= (1/16)(1 + e_1248 + e_2358 + e_3468 + e_4578 + e_5618 + e_6728 + e_7138
- e_3567 - e_4671 - e_5712 - e_6123 - e_7234 - e_1345 - e_2456 + e_J)
with four factors [and where J = 12345678] ...
The division ring F = f Cl(p,q) f = { PSI in V | PSI f = f PSI }
is isomorphic to R, C, or H

according as $p - q = 0, 1, 2, \mod 8$, $p - q = 3 \mod 4$, or $p - q = 4, 5, 6 \mod 8$. ...".
In "Idempotent Structure of Clifford Alghebras" (Acta Applicandae Mathematicae 9 (1987) 165-173) Pertti Lounesto and G. P. Wene said:
"... An idempotent e is primitive if it is not a sum of two nonzero annihilating idempotents and minimal if it is a minimal element in the set of all nonzero idempotents with order relation $f \leq e$ if and only if $ef = f = fe$.
These last two properties of an idempotent e are equivalent. An idempotent e is primitive if e is the only nonzero idempotent of the subring eAe.
A subring S of A is a left ideal if ax is in S for all a in A and x in S.
A left ideal is minimal if it does not contain properly any nonzero left ideals.
... if S is a minimal left ideal of A,
then either $Ss = 0$ or $S = Ae$ for some idempotent e.
Spinor spaces are minimal left ideals of a Clifford algebra.
Any minimal left ideal S of a Clifford algebra $A = Rp,q$ is of the form $S = Ae$ for some primitive idempotent e of Rp,q.
... if e is a primitive idempotent of Rp,q then

$$\begin{matrix} e & 0 \\ 0 & 0 \end{matrix}$$

is a primitive idempotent of $Rp,q(2) = Rp+1,q+1$
... The maximum number of mutually annihilating primitive idempotents in the Clifford algebra Rp,q is 2^k where k = q - r_q-p .
...[where]... r_i ...[is the]... Radon-Hurwitz number ...
These mutually annihilating primitive idempotents sum up to 1.
If mutually annihilating primitive idempotents sum up to 1,
then in a simple ring, such a sum has always the same number of summands.
... Lattices Generated by Idempotents
A lattice is a partially ordered set where each subset of two elements has a least upper bound and a greatest lower bound. Any set of idempotents of a ring A is partially ordered under the ordering defined by $e \leq f$ if and only if $ef = e = fe$.
If e and f are commuting idempotents, then ef and $e + f - ef$ are, respectively, a greatest lower bound and a least upper bound relative to the partial ordering defined. Hence, any set of commuting idempotents generate a lattice.
This lattice is complemented and distributive.
...
Let e1 , e2 , ... , es in Rp,q be a set of mutually annihilating primitive idempotents summing up to 1. Then the set e1 , e2 , ... , es generates a complemented and distributive lattice of order 2^s , where s = 2^k , k = q - r_q-p
...
EXAMPLE [I have changed the example from R3,1 to R0,8 and paraphrased]
In the Clifford algebra R0,8 = R(16) we have k = 8 - r_8 = 8 - 4 = 4

and so primitive idempotents can have 4 commuting factors of type (1/2)(1 + eT) . Furthermore s = 2^k = 16 and so R0,8 can be represented by 16x16 matrices R(16), and there are 2^s = 2^16 = 65,536 commuting idempotents in the lattice generated by the 16 mutually annihilating primitive idempotents ...
this lattice looks like ... a 16-dimensional analogy of the cube ...".

The Clifford algebra R0,8 = Cl(0,8) is 2^8 = 16x16 = 256-dimensional with graded structure such that it
is represented by the geometric structure of a simplex.

The Spinors of R0,8 = Cl(0,8) are sqrt(256) = 16-dimensional with no simplex-type graded structure so that it
is represented by the geometric structure of a cube.

248-dim E8 = 120-dim Cl(16) bivectors + 128-dim Cl(16) half-spinors and Cl(16) = Cl(8) x Cl(8)
so the structure of the 128-dim Cl(16) half-spinors is important for E8 Physics.

The Clifford algebra Cl(16) (also denoted R0,16) is the real 256x256 matrix algebra R(256) for which we have k = 16 - r_16 = 16 - 8 = 8
and so primitive idempotents can have 8 commuting factors of type (1/2)(1 + eT) . Furthermore s = 2^k = 256 and so R0,16 can be represented by 256x256 matrices R(256), and there are 2^s = 2^256 = 1.158 x 10^77 commuting idempotents in the lattice generated by the 256 mutually annihilating primitive idempotents.

E8 lives in Cl(16) as
248-dim E8 = 120-dim bivectors of Cl(16) + 128-dim half-spinor of Cl(16).
Since Cl(16) bivectors are all in one grade of Cl(16)
and Cl(16) half-spinors have no simplex-type graded structure
E8 does not get detailed graded structure from Cl(16) gradings,
but only the Even-Odd grading obtained by
splitting 128-dim half-spinor into two mirror image 64-dim parts:
E8 = 64 + 120 + 64

E8 has only a Cl(16) half-spinor so there are in E8 Physics 2^(s/2) = 2^128 commuting idempotents in the lattice generated by the 128 mutually annihilating primitive idempotents. 2^128 = about 3.4 x 10^38 the square root of which is about the ratio (Hadron mass / Planck mass)^2 of the Effective Mass Factor for Gravity strength.

The typical Hadron mass can be thought of in terms of superposition of Pions:

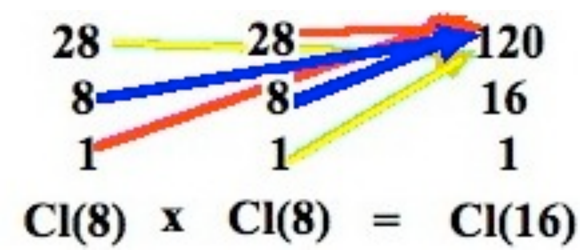
In E8 Physics, at a single spacetime vertex, a Planck-mass black hole is the Many-Worlds quantum sum of all possible virtual first-generation particle-antiparticle fermion pairs permitted by the Pauli exclusion principle to live on that vertex.Once a Planck-mass black hole is formed, it is stable in in E8 Physics. Less mass would not be gravitationally bound at the vertex. More mass at the vertex would decay by Hawking radiation.Since Dirac fermions in 4-dimensional spacetime can be massive (and are massive at low enough energies for the Higgs mechanism to act), the Planck mass in 4-dimensional spacetime is the sum of masses of all possible virtual first-generation particle-antiparticle fermion pairs permitted by the Pauli exclusion principle. A typical combination should have several quarks, several antiquarks, a few colorless quark-antiquark pairs that would be equivalent to pions, and some leptons and antileptons. Due to the Pauli exclusion principle, no fermion lepton or quark could be present at the vertex more than twice unless they are in the form of boson pions, colorless first-generation quark-antiquark pairs not subject to the Pauli exclusion principle. Of the 64 particle-antiparticle pairs, 12 are pions. A typical combination should have about 6 pions.
If all the pions are independent, the typical combination should have a mass of 0.14x6 GeV = 0.84 GeV. However, just as the pion mass of 0.14 GeV is less than the sum of the masses of a quark and an antiquark, pairs of oppositely charged pions may form a bound state of less mass than the sum of two pion masses. If such a bound state of oppositely charged pions has a mass as small as 0.1 GeV, and if the typical combination has one such pair and 4 other pions, then the typical combination should have a mass in the range of 0.66 GeV so that

sqrt(3.4 x 10^38) = 1.84 x 10^19
while Planck Mass = 1.22 x 10^19 GeV = 1.30 x 10^19 Proton Mass =
= 1.85 x 10^19 Hadron Mass

**In terms of the Graded Structure of Cl(16)
the 256 Cl(16) Primitive Idempotents can be understood
in terms of graded structures of the Cl(8) and E8 substructures of Cl(16):**

**The detailed E8 graded structure 8 + 28 + 56 + 64 + 56 + 28 + 8
comes from the grades of the Cl(8) factors of Cl(16) = Cl(8)xCl(8).**

The Even 120 of E8 breaks down in terms of Cl(8) factors as



120 = 1x28 + 8x8 + 28x1 = 28 + 64 + 28

The Odd 128 = 64 + 64 breaks down as

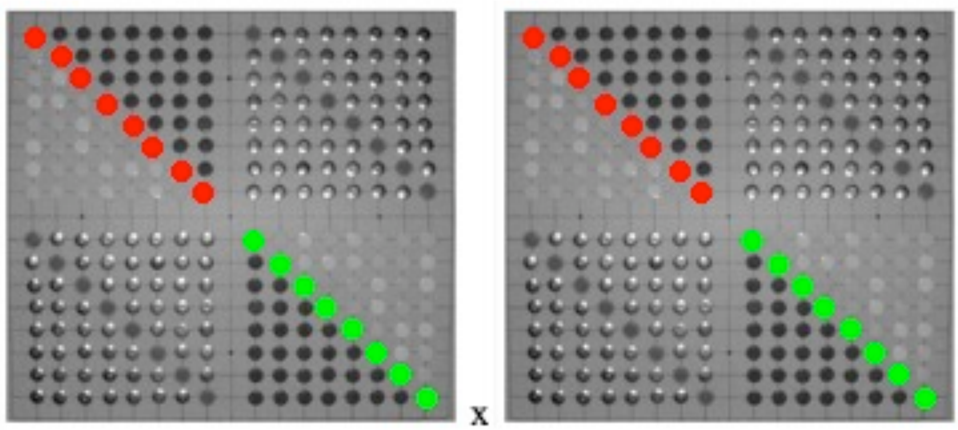
Spinors:

$$(8s+8c) \times (8s+8c) = \begin{array}{c} (8s \times 8s + 8c \times 8c) \\ + \\ (8s \times 8c + 8c \times 8s) \end{array}$$

**to become
64 + 64 = 8 + 56 + 56 + 8**

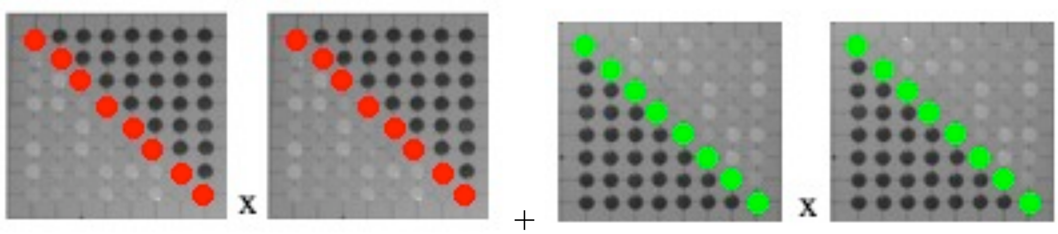
Here are some details about the half-spinors of E8:

The +half-spinors (red) and -half-spinors (green) of Cl(8) are the 8+8 = 16 diagonal entries of the 16x16 real matrix algebra that is Cl(8), so that Cl(16) = Cl(8) x Cl(8) can be represented as:



and

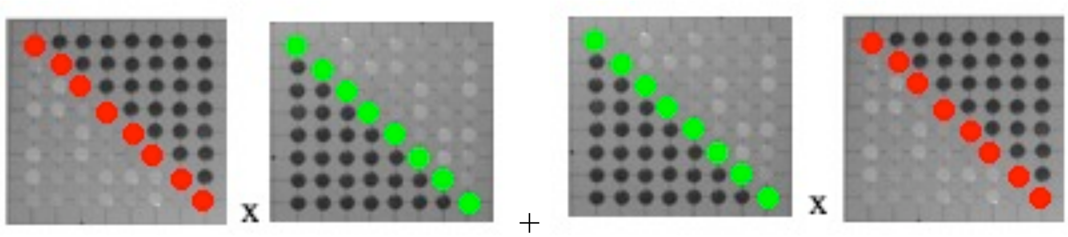
the 16x16 = 256 spinors of Cl(16) (the diagonal entries of R(256)) can be represented as the sum of the diagonal product terms



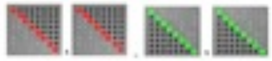
64+64 = 128

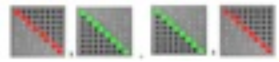
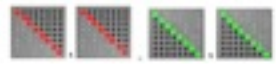
(these two (pure red and pure green) are the Cl(16) +half-spinor which decomposes physically into particles (red) and antiparticles (green))

+

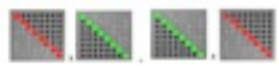
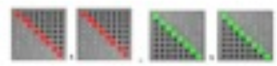


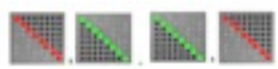
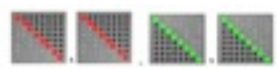
64+64 = 128

(these two (mixed red and green) are the Cl(16) -half-spinor which do not decompose readily into particles (red) and antiparticles (green)) grade-0: 1 PurePI 

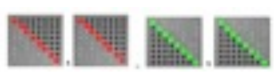
grade-1: 16 NotPI
grade-2: 120 NotPI
grade-3: 560 NotPI
grade-4: 1820 = 1792 + 14 MixedPI 
+ 14 PurePI 

grade-5: 4368 NotPI
grade-6: 8008 NotPI
grade-7: 11440 NotPI

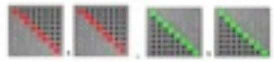
grade-8: 12870 = 12672 + 100 MixedPI 
+ 98 PurePI 

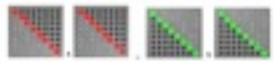
grade-9: 11440 NotPI
grade-10: 8008 NotPI
grade-11: 4368 NotPI
grade-12: 1820 = 1792 + 14 MixedPI 
+ 14 PurePI 

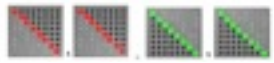
grade-13: 560 NotPI
grade-14: 120 NotPI
grade-15: 16 NotPI

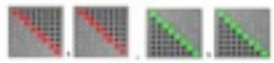
grade-16: 1 PurePI 

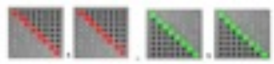
Only the PurePI Cl(16) +half-spinor has scalar grade-0 and pseudoscalar grade-16

grade-0: 1 PurePI 

grade-4: 14 PurePI 

grade-8: 98 PurePI 

grade-12: 14 PurePI 

grade-16: 1 PurePI 

so it is the only half-spinor that can physically represent a Higgs scalar
and is the only half-spinor in the E8 of E8 Physics.

Further, for E8 to describe a consistent E8 Physics model, it must be that
E8 = Cl(16) bivectors + Cl(16) +half-spinor
with physical distinction between particles and antiparticles
and that
E8 does not contain the Cl(16) -half-spinor made up of particle/antiparticle mixtures.
In the context of physics models,
the Cl(16) -half-spinors correspond to fermion antigenerations that are not realistic
and their omission from E8 allows E8 Physics to be chiral and realistic.

E8 with graded structure 8 + 28 + 56 + 64 + 56 + 28 + 8 lives in Cl(16)
as
248-dim E8 = 120-dim bivectors of Cl(16) + 128-dim half-spinor of Cl(16).

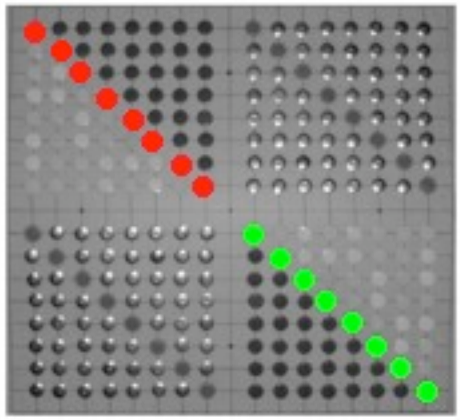
The two half-spinors of Cl(16) are Left Ideals of a Cl(16) Primitive Idempotent.

Due to 8-periodicity of Real Clifford Algebras Cl(16) = Cl(8) x Cl(8)
where x is tensor product. Let Primitive Idempotent be denoted by PI
and J = 12345678 :

Cl(16)PI = Cl(8)PI x Cl(8)PI

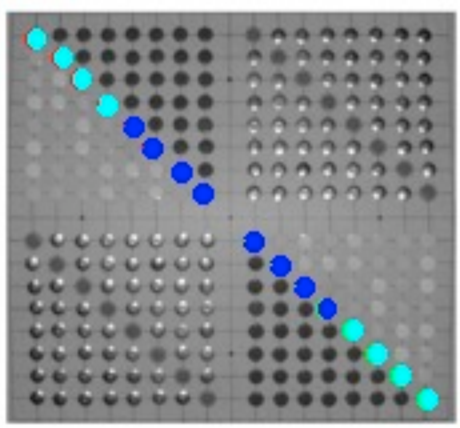
Cl(8)PI = (1/16) (1 + e_1248) (1 + e_2358) (1 + e_3468) (1 + e_4578) =

= (1/16)(1
+ e_1248 + e_2358 + e_3468 + e_4578 + e_5618 + e_6728 + e_7138



- e_3567 - e_4671 - e_5712 - e_6123 - e_7234 - e_1345 - e_2456
+ e_J) =

= (1/16)(
1 +
+ e_1248 + e_2358 + e_3468
- e_3567 - e_4671 - e_5712
+ e_J



+ e_4578 + e_5618 + e_6728 + e_7138
- e_6123 - e_7234 - e_1345 - e_2456
)

256-dim Cl(8) has graded structure 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1
16-dim Cl(8)PI has graded structure 1 + 14 + 1 = 1 + (8+6) + 1
16-dim Cl(8)PI = 8-dim Cl(8)PIE8 + 8-dim Cl(8)PInotE8
where
8-dim Cl(8)PIE8 has graded structure of only 8 in the middle grade
plus
8-dim Cl(8)PInotE8 has graded structure 1 + 6 + 1
8-dim Cl(8)PIE8 is contained in the middle 64 of E8 graded structure
8 + 28 + 56 + 64 + 56 + 28 + 8
so that
since the physical interpretation of the middle 64 is
8 momentum components of 8-dim position spacetime
the 8-dim Cl(8)PIE8 corresponds to a one-component field over 8-dim spacetime
and
therefore Cl(8)PIE8 describes a scalar field over 8-dim spacetime
and so a Higgs field in E8 Physics spacetime.

8-dim Cl(8)PInotE8 with graded structure 1 + 6 + 1
corresponds to the part of Cl(8)PI that is in Cl(8) but not in E8
so that

Cl(8) with graded structure 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1
=
Cl(8)PInotE8 with graded structure 1 + 6 + 1
+
E8 with graded structure 8 + 28 + 56 + 64 + 56 + 28 + 8

and
therefore Cl(8)PInotE8 describes the Clifford algebra structure beyond E8
(1 scalar and 6 middle-grade and 1 pseudoscalar)
that produces the half-spinors that belong to E8
and
therefore describes the coupling between the Higgs field and half-spinor Fermions.

The Higgs-Fermion coupling, below the freezing out of a preferred Quaternionic substructure of 8-dim Octonionic E8 Physics spacetime, produces
the Mayer Mechanism Higgs field of 8-dim Batakis Kaluza-Klein spacetime.

The Higgs-Fermion coupling, below ElectroWeak Symmetry Breaking Energy, gives mass to Fermions.

Since the 128-dim half-spinor part of E8 comes from
Cl(16)PI = Cl(8)PI x Cl(8)PI
the E8 Higgs-Fermion is based on
two copies (one from each Cl(8)PI factor) of a scalar Higgs field over spacetime

so that

two copies of Cl(8)PIE8 show that the E8 Physics Higgs field is a scalar doublet.

As Cottingham and Greenwood said in their book "An Introduction to the Standard Model of Particle Physics" (2nd ed, Cambridge 2007):
"... Higgs ... mechanism ...[uses]... a complex scalar field ... [i]n place of [which]... we [can] have two coupled real scalar fields ...".

As Steven Weinberg said in his book "The Quantum Theory of Fields, v. II" (Cambridge 1996 at pages 317-318 and 356):
"... With only a single type of scalar doublet, there is just one ... term that satisfies SU(2) and Lorentz invariance ... At energies below the electroweak breaking scale, this yields an effective interaction ... this gives lepton number non-conserving neutrino masses at most of order (300 GeV)^2 / M ... For instance,
in the so-called see-saw mecanism, a neutrino mass of this order
would be produced by exchange of a heavy neutral lepton of mass M ...
M is expected to be of order 10^15 - 10^18 GeV,
so we would expect neutrino masses in the range 10^(-4) - 10^(-1) ...
A similar analysis shows that there are interactions of dimensionality six
that violate both baryon and lepton number conservation, involving
three quark fields and one lepton field. Such interactions would have
coupling constants of order M^(-2), and would lead to processes like
proton decay, with rates proportional to M^(-4). ...".

and

the part of the Cl(16) Primitive Idempotent that is not in the E8 in Cl(16)
is the product Cl(8)PInotE8 x Cl(8)PInotE8 of two copies of Cl(8)PInotE8
each copy having graded structure 1 + 6 + 1 (grades 0 and 4 and 8)
so that
the part of the Cl(16) Primitive Idempotent that is not in the E8 in Cl(16)
has graded structure 1 + 12 + 38 + 12 + 1 (grades 0 and 4 and 8 and 12 and 16).
The total dimension of those Cl(16) grades are:
1 and 1820 and 128870 and 1820 and 1.

Cl(8) **256 = 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1**

Primitive **16 = 1** **+ 6** **+ 1**
Idempotent **+ 8**

E8 Root Vectors 240 = 8 + 28 + 56 + 56 + 56 + 28 + 8

Greg Trayling and W. E. Baylis in Chapter 34 of "Clifford Algebras - Applications to Mathematics, Physics, and Engineering", 2004, Proceedings of 2002 Cookeville Conference on Clifford Algebras, ed. by Rafal Ablamowicz
(see also hep-th/0103137) said:
"... the exact gauge symmetries U(1)Y x SU(2)L x SU(3)C of the minimal standard model arise ...[from]... symmetries of ... a ... space with ... four extra spacelike dimensions ...
[compare the Batakis M4xCP2 4+4=8-dimensional Kaluza-Klein model]...
Rather than embed the gauge broups into some master group, we infix the Dirac algebra into the ... Clifford algebra Cl(7) ...[in which]... the unit vectors
e1 ,e2 , ... , e7 are chosen to represent ... spacelike directions ...
We further choose e1 , e2 , e3 to represent ... physical space and ...
e4 , e5 , e6 , e7 to ... represent ... four ...dimensions ... orthogonal to physical space ... [compare the Cl(8) of E8 Physics which is represented by 16x16 matrices with two 8-dimensional half-spinor spaces and in which the 8 unit vectors
e0 , e1 ,e2 , ... , e7 represent Batakis 8-dimensional spacetime M4xCP2 where
e0 , e1 ,e2 , e3 represents M4 and e4 , e5 ,e6 , e7 represents CP2]...
To describe one generation of the standard model, we use the algebraic spinor PSI in Cl(7) ... there are eight independent primitive idempotents that can each be used to reduce PSI to a spinor representing a fermion doublet ...
Each of the eight ... primitive idempotents ... projects PSI onto one of eight minimal left ideals of Cl(7) ...
[compare the 8+8 = 16 primitive idempotents of Cl(8) which correspond to 8 first-generation fermion particles and their 8 antiparticles] ...
we previously disregarded the higher-dimensional vector components ... This ... vector space ... then ... affords a natural inclusion of the minimal Higgs field ...
The Higgs field ... arises here simply as a coupling to the higher-dimensional vector components ...".

[compare the E8 Physics model relationship between the Higgs and the Cl(8) primitive idempotents which live in grades 0 and 4 and 8 of Cl(8)]

Klaus Dietz in arXiv quant-ph/0601013 said:
"... **m-Qubit states are embedded in Cl(2m) Clifford algebra**. ...
This ... allows us to arrange the 2^(2m) - 1 real coordinates of a m-Qubit state in multidimensional arrays which are shown to 'transforn\m' as O(2m) tensors ...
A hermitian 2^m x 2^m matrix requires 2^(2m) real numbers for a complete parameterization. Thus m-qubit states can be expanded in terms of I and the products introduced. Clifford numbers are the starting point for the construction of a basis in R-linear space of hermitian matrices:
this basis is construed as a Clifford algebra Cl(2m) ...".

Stephanie Wehner in arXiv 0806.3483 said:
"... A Clifford algebra of n generators is isomorphic to a ... algebra of matrices of size 2^(n/2) x 2^(n/2) for n even ...
we can view the operators G1 , ... , G2n as 2n orthogonal vectors forming a basis for a 2n-dimensional real vector space R2n ...
each operator Gi has exactly two eigenvalues +/-1 ...
we can express each Gi as Gi = G0i - G1i
where G0i and G1i are projectors onto the positive and negative eigenspace of Gi
... for all i,j with i =/= j   Tr(GiGj) = (1/2)Tr(GiGj + GjGi) = 0
that is all such operators are orthogonal with respect to the Hilbert-Schmidt inner product ... the collection of operators
1
Gj   (1 ≤ j ≤ 2n)
Gjk := iGjGk   (1 ≤ j < k ≤ 2n)
Gjkl := GjGkGl   (1 ≤ j < k < l ≤ 2n)
...
G12...(2n) := iG1G2 ... G2n =: G0
forms an orthogonal basis for ... the d x d matrices ... with d = 2^n ...
We saw ... how to construct such **a basis ... based on mutually unbiased bases ... the well-known Pauli basis, given by the 2^(2n) elements of the form**
Bj = B1j x ...[tensor product]... x Bnj with Bij in { I , sx , sy , sz } ...
we obtain a whole range of ... statements as we can find different sets of 2n anti-commuting matrices within the entire set of 2^(2n) basis elements ...
the subspace spanned by the elements G1 , ... , G2n plays a special role ...
when considering the state minimizing our uncertainty relation,
only the 1-vector coefficients play any role. The other coefficients do not contribute at all to the minimization problem. ...
Anti-commuting Clifford observables obey the strongest possible uncertainty relation for the von Neumann entropy: if we have no uncertainty for one of the measurements, we have maximum uncertainty for all others. ...".

Monique Combescure in quant-ph/060509, arXiv 0710.5642 and 0710.5643 said:
"... two basic unitary d x d matrices U , V ... constructed by Schwinger ... q := exp (2 i pi / d) ... are of the following form:

$$U := \mathrm{Diag}(1, q, q^2, ..., q^{d-1})$$

$$V := \begin{pmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 \\ 1 & 0 & 0 & . & . & 0 \end{pmatrix}$$

... the matrices U and V are called
"generalized Pauli matrices on d-state quantum systems" ...
U, V generate the discrete Weyl-Heisenberg group ... U, V allows to find MUB's ... in dimension d there is at most d+1 MUB, and exactly d + 1 for d a prime number ...
A d x d matrix C is called circulant ... if all its rows and columns are successive circular permutations of the first ... the theory of circulant matrices allows to recover the result that there exists p + 1 Mutually Unbiased Bases in dimension p, p being a... prime number ... Then the MUB problem reduces to exhibit a circulant matrix C which is a unitary Hadamard matrix, such that its powers are also circulant unitary Hadamard matrices. Then using Discrete Fourier Transform Fd which diagonalizes all circulant matrices, we have shown that a MUB in that case is just provided by the set of column vectors of the set of matrices
{ Fd, 1, C, C2, ... , C(d-1) }
...
the theory of block-circulant matrices with circulant blocks allows to show ...
that if d = p^n (p a prime number, n any integer)
there exists d + 1 mutually Unbiased Bases in Cd ...".

Stephen Brierley, Stefan Weigert, and Ingemar Bengtsson in arXiv 0907.4097 said:
"... All complex Hadamard matrices in dimensions two to five are known ...
In dimension three there is ... only one dephased complex Hadamard matrix up to equivalence. It is given by the (3 x 3) discrete Fourier matrix

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

defining w = exp(2 pi i / 3)
...
In dimension d = 4 , all 4 x 4 complex Hadamard matrices are equivalent to a member of the ... one-parameter family of complex Hadamard matrices ...

$$F_4(x) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & ie^{ix} & -ie^{ix} \\ 1 & -1 & -ie^{ix} & ie^{ix} \end{pmatrix}, \quad x \in [0, \pi]$$

... There is one three-parameter family of triples ...
Only one set of four MU bases exists ...
there is a unique way to a construct five MU bases which is easily seen to be equivalent to the standard construction of a complete set of MU bases ... d = 4 ...

d	2	3	4	5	6
pairs	1	1	∞^1	1	$\geq \infty^3$
triples	1	1	∞^3	2	$\geq \infty^2$
quadruples	-	1	1	1	?
quintuples	-	-	1	1	?
sextuples	-	-	-	1	?

... The notion of equivalence used in this paper ... is mathematical in nature ...
Motivated by experiments, there is a finer equivalence of complete sets of MU bases based on the entanglement structure of the states contained in each basis ...
For dimensions that are a power of two, a complete set of MU bases can be realized using Pauli operators acting on each two-dimensional subsystem.
Two sets of MU bases are then called equivalent when they can be factored into the same number of subsystems. For d = 2, 4 this notion of equivalence also leads to a unique set of (d + 1) MU bases. However, for d = 8, 16, . . . complete sets of MU bases can have different entanglement structures even though they are equivalent up to an overall unitary transformation ...".

P. Dita in arXiv 1002.4933 said:
"... Mutually unbiased bases (MUBs) constitute a basic concept of quantum information ... Its origin is in the Schwinger paper ... "Unitary operator bases", Proc.Nat. Acad. Sci.USA, 46 570-579 (1960) ...
Two orthonormal bases in Cd, A = (a1, . . . , ad) and B = (b1, . . . , bd), are called MUBs if ... the product A B* of the two complex Hadamard matrices generated by A and B is again a Hadamard matrix, where * denotes the Hermitian conjugate ...
The technique for getting MUBs for p prime was given by Schwinger ... who made use of the properties of the Heisenberg-Weyl group
...
[in this paper] An analytical method for getting new complex Hadamard matrices by using mutually unbiased bases and a nonlinear doubling formula is provided. The method is illustrated with the n = 4 case that leads to a rich family of eight-dimensional Hadamard matrices that depend on five arbitrary phases ... The ... matrices are new ... the only [prior] known result parametrized by five phases is the [n = 8] complex Hadamard matrix stemming from the Fourier matrix F8
...
real Sylvester-Hadamard matrices ...[have a]... solution for n = 8 ...

$$S = \begin{bmatrix} a & b & c & d & l & m & n & p \\ b & -a & -d & c & m & -l & p & -n \\ c & d & -a & -b & n & -p & -l & m \\ d & -c & b & -a & p & n & -m & -l \\ l & -m & -n & -p & -a & b & c & d \\ m & l & p & -n & -b & -a & d & -c \\ n & -p & l & m & -c & -d & -a & b \\ p & n & -m & l & -d & c & -b & -a \end{bmatrix}$$

... for real Hadamard matrices with dimension d = 2, 4, 8, 12 there is only one matrix under the usual equivalence ... there is an other type of matrix equivalence ... two matrices ... are equivalent if and only if they have the same spectrum ... However a simple spectral computation of the h1, h2, h3, h4 matrices shows that only the matrices h1 and h3 are equivalent, and h1 is not equivalent to h2 and h4, nor h2 is equivalent to h4 ...[so that]... we do not suggest the use of the new equivalence ... for real Hadamard matrices ... because it will cause dramatic changes in the field ...".

Standard Model Higgs compared to E8 Physics Higgs

The conventional Standard Model has structure:
spacetime is a base manifold;
particles are representations of gauge groups
  gauge bosons are in the adjoint representation
  fermions are in other representations (analagous to spinor)
  Higgs boson is in scalar representation.

E8 Physics (see vixra 1108.0027 and tony5m17h.net) has structure
(from 248-dim E8 = 120-dim adjoint D8 + 128-dim half-spinor D8):
spacetime is in the adjoint D8 part of E8 (64 of 120 D8 adjoints)
gauge bosons are in the adjoint D8 part of E8 (56 of the 120 D8 adjoints)
fermions are in the half-spinor D8 part of E8 (64+64 of the 128 D8 half-spinors.

There is no room for a fundamental Higgs in the E8 of E8 Physics.
However,
for E8 Physics to include the observed results of the Standard Model
it must have something that acts like the Standard Model Higgs
even though it will NOT be a fundamental particle.

To see how the E8 Physics Higgs works,
embed E8 into the 256-dimensional real Clifford algebra Cl(8):

Cl(8) 256 = 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1

Primitive 16 = 1 + 6 + 1
Idempotent + 8

E8 Root Vectors 240 = 8 + 28 + 56 + 56 + 56 + 28 + 8

The Cl(8) Primitive Idempotent is 16-dimensional and can be decomposed
into two 8-dimensional half-spinor parts each of which is related by Triality
to 8-dimensional spacetime and has Octonionic structure. In that decomposition:
the 1+6+1 = (1+3)+(3+1) is related to two copies of
a 4-dimensional Associative Quaternionic subspace of the Octonionic structure
and
the 8 = 4+4 is related to two copies of
a 4-dimensional Co-Associative subspace of the Octonionic structure
(see the book “Spinors and Calibrations” by F. Reese Harvey)

The 8 = 4+4 Co-Associative part of the Cl(8) Primitive Idempotent
when combined with the 240 E8 Root Vectors
forms the full 248-dimensional E8.
It represents a Cartan subalgebra of the E8 Lie algebra.

The (1+3)+(3+1) Associative part of the Cl(8) Primitive Idempotent is the Higgs of E8 Physics.

The half-spinors generated by the E8 Higgs part of the Cl(8) Primitive Idempotent represent:

neutrino; red, green, blue down quarks; red, green, blue up quarks; electron

so
the E8 Higgs effectively creates/annihilates the fundamental fermions and
the E8 Higgs is effectively a condensate of fundamental fermions.

In E8 Physics the high-energy 8-dimensional Octonionic spacetime reduces,
by freezing out a preferred 4-dim Associative Quaternionic subspace,
to a 4+4 -dimensional Batakis Kaluza-Klein of the form M4 x CP2
with 4-dim M4 physical spacetime.

Since the (1+3)+(3+1) part of the Cl(8) Primitive Idempotent
includes the Cl(8) grade-0 scalar 1
and 3+3 = 6 of the Cl(8) grade-4 which act as pseudoscalars for 4-dim spacetime
and the Cl(8) grade-8 pseudoscalar 1
the E8 Higgs transforms with respect to 4-dim spacetime as a scalar (or pseudoscalar) and in that respect is similar to Standard Model Higgs.

Not only does the E8 Higgs fermion condensate transform with respect
to 4-dim physical spacetime like the Standard Model Higgs but
the geometry of the reduction from 8-dim Octonionic spacetime to 4+4 -dimensional Batakis Kaluza-Klein, by the Mayer mechanism, gives E8 Higgs the ElectroWeak Symmetry-Breaking Ginzburg-Landau structure.

Since the second and third fermion generations emerge dynamically from the
reduction from 8-dim to 4+4 -dim Kaluza-Klein, they are also created/annihilated
by the Primitive Idempotent E8 Higgs and are present in the fermion condensate.
Since the Truth Quark is so much more massive that the other fermions,
the E8 Higgs is effectively a Truth Quark condensate.
When Triviality and Vacuum Stability are taken into account,
the E8 Higgs and Truth Quark system has 3 mass states.

Since it creates/annihilates Fermions,
the (1+3)+(3+1) Associative part of the Cl(8) Primitive Idempotent
is a Fermionic Condensate Higgs structure.
The creation/annihilation operators have graded structure similar to part of a Heisenberg algebra

$$64 + 0 + 64$$

Since it creates/annihilates the 8-dimensional SpaceTime
represented by the Cartan Subalgebra of the E8 Lie Algebra,
the 8 = 4+4 Co-Associative part of the Cl(8) Primitive Idempotent
is a Bosonic Condensate Spacetime structure.
The creation/annihilation operators correspond to position-momentum related by Fourier Transform and to an 8x8 = 64-dimensional U(8)

E8 has two D4 Lie subalgebras D4 and D4* related by Fourier Transform:

> 28-dimensional D4 acting on M4 4-dim Physical SpaceTime and containing a Spin(2,4) subalgebra for Conformal MacDowell-Mansouri Gravity;
> and
> 28-dimensional D4* acting on CP2 Internal Symmetry Space and containing a U(4) subalgebra for the Batakis Standard Model gauge groups.

Taken together, the D4 and U(8) and D4* have graded structure

$$28 + 64 + 28$$

that breaks down into a semi-simple 63-dimensional SU(8)

$$63$$

and a Heisenberg Algebra

$$28 + 1 + 28$$

When the Fermionic 64 + 0 + 64 is added, the Heisenberg Algebra becomes

$$92 + 1 + 92$$

and the total 92 + U(8) + 92 is seen to be the contraction of E8 into the semidirect product of semisimple SU(8) and Heisenberg Algebra 92 + U(1) + 92

Robert Hermann in “Lie Groups for Physicists” (Benjamin 1966) said:
“... Let G be a Lie group ... imbed G into the associative algebra U(G) ... the universal ... enveloping algebra ...
the “polynomials” of the .. basis [elements] of G ... form a basis for U(G) ...
the center of U(G) ...[is]... the Casimir operators of G ...[whose]... number ...[is]... equal to ... the dimension of its Cartan subalgebras ...
every polynomial ... invariant under AdG ... arise[s] ... from a Casimir operator ...
when G is semisimple, Ad G acting on G admits an invariant polynomial of degree 2 ... the Killing form ... This is the simplest such Casimir operator
...
there is a group-theoretical construction which in certain situations reduces to the Fourier transform. To describe it, we need ... a Lie group G, two subgroups L and H of G, and linear representations ... of L and H ... on a vector space U, which determines vector bundles E and E’ over G/L and G/H. ...
A cross section PSI of ... E’ over G/H is an eigenvector of each Casimir operator of U(G) .... its transform PSI*, considered as a function on G/K, is also an eigenfunction of each Casimir operator of U(G). ...”.

Rutwig Campoamor-Stursberg in “Contractions of Exceptional Lie Algebras and SemiDirect Products” (Acta Physica Polonica B 41 (2010) 53-77) said:
“... it is of interest to analyze whether ... semidirect products ... of semisimple and Heisenberg Lie algebras ... appear as contractions of semisimple Lie algebras ...
Let s be a ... semisimple Lie algebra. For the indecomposable semidirect product g = s + Hn the number of Casimir operators is given by N(g) = rank(s) + 1
... In some sense, the Levi subalgebra s determines these Casimir invariants, to which the central charge (the generator of the centre of the Heisenberg algebra) is added. ... the quadratic Casimir operator will always contract onto the square of the centre generator of the Heisenberg algebra ...
... We have classified all contractions of complex simple exceptional Lie algebras onto semidirect products ... s + h_N ... of semisimple and Heisenberg algebras.
An analogous procedure holds for the real forms of the exceptional algebras ...
Contractions of E8 ... E8 contains D8 contains A7 ...[and for E8]... N = 92
... This reduction gives rise to the contraction ...[E8 to A7 + H92]...
E8 ... has primitive Casimir operators ... of degrees ...[2,8,12,14,18,20,24,30]...
D8 ... has primitive Casimir operators ... of degrees ...[2,4,6,8,10,12,14,8]...
A7 ... has primitive Casimir operators ... of degrees ...[2,3,4,5,6,7,8]...”.

The E8 primitive Casimirs 2 ,8, 12, 14, 18, 20, 24, 30 contract as follows:

2 to the center U(1) of H92.
8, 12, 14 to the 8, 12, 14 of D8 and to the 4=8/2, 6=12/2, 7=14/2 of A7

18, 20, 24, 30 to the 4=18-14, 6=20-14, 10=24-14, 8=(1/2)(30-14) of D8
and to the 2=4/2, 3=6/2, 5=10/2, 8 of A7

The 2, 8, 12, 14 of E8 are dual to the 30, 24, 20, 18 of E8 such that
2+30 = 8+24 = 12+20 = 14+18 = 32.

The E8 primitive Casimirs correspond to the Cartan subalgebras of E8 and of D8
and also to 8-dim Spacetime and 4+4-dim Batakis Kaluza-Klein M4 x CP2

The 2, 8, 12, 14 Casimirs of E8 correspond to
the (1+3)-dim M4 Batakis Physical Spacetime

The 18, 20, 24, 30 Casimirs of E8 correspond to
the 4-dim CP2 Batakis Internal Symmetry Space

Weyl Symmetric Polynomial Degrees and Topological Types:

E8:
degrees - 2, 8, 12, 14, 18, 20, 24, 30
note that 1, 7, 11, 13, 17, 19, 23, and 29 are all relatively prime to 30
type - 3, 15, 23, 27, 35, 39, 47, 59; center = Z1 = 1 = trivial

D8 Spin(16):
degrees - 2, 4, 6, 8, 10, 12, 14, 8
type - 3, 7, 11, 15, 19, 23, 27, 15; center = Z2 + Z2

A7 SU(8):
degrees - 2, 3, 4, 5, 6, 7, 8
type - 3, 5, 7, 9, 11, 13, 15; center = Z8

Luis J. Boya has written a beautiful paper “Problems in Lie Group Theory” math-ph/0212067 and here are a few of the interesting things he says:

"... Given a Lie group in a series G(n) ... how is the group G(n+1) constructed?

For the **orthogonal series (Bn and Dn)** ... given O(n) acting on itself, that is, the adjoint (adj) representation, and the vector representation, n, ...
Adj O(n) + Vect O(n) -> Adj O(n+1) ...

For the unitary series SU(n) ... Adj SU(n) + Id + n + n* = Adj SU(n+1) ...

For the symplectic series
Sp(n) = Cn ... Adj Sp(n) + Adj Sp(1) + 2(n + n*) = Adj Sp(n+1) ...

For **G2** ... **Adj SU(3) + n + n* -> G2** ...[in addition, I conjecture the existence of an alternate construction: **Adj O(4) + Vect O(4) + Spin O(4) = G2** ,
where Spin O(4) is its Spin representation, a notation that I will continue to use in the rest of this quotation instead of the notation Spin(4) that Boya uses, because I want to reserve the notation Spin(4) for the covering group of SO(4). Note that Spin O(n) for even n is reducible to two copies of mirror image half-spinor representations half-Spin O(n)]...

For the **exceptional groups, the F4 & E series** ...

- **Adj SO(9) + Spin O(9) -> Adj F4 (36+16=52)**
- **Adj SO(10) + Spin O(10) + Id -> Adj E6 (45+32+1=78)**
- **Adj SO(12) + Spin O(12) + Sp(1) -> Adj E7 (66+64+3=133)**
- **Adj SO(16) + [half-]Spin O(16) -> Adj E8 ([120+128=248])**

Notice that 8+1 , 8+2 , 8+4 , and 8+8 appear. In this sense the octonions appear as a "second coming " of the reals, completed with the spin, not the vector irrep. ...
This confirms that the F4 E6-7-8 corresponds to
the octo, octo-complex, octo-quater and octo-octo birings,
as the Freudenthal Magic Square confirms. ...
Another ... question ... is the geometry associated to the exceptional groups ...
Are we happy with G2 as the automorphism group of the octonions, F4 as the isometry of the [octonion] projective plane, E6 (in a noncompact form) as the collineations of the same, and E7 resp. E8 as examples of symplectic resp. metasymplectic geometries? ... one would like to understand the exceptional groups ... as automorphism groups of some natural geometric objects. ...

The gross topology of Lie groups is well-known. The non-compact case reduces to compact times an euclidean space (Malcev-Iwasawa). The compact case is reduced to a finite factor, a Torus, and a semisimple compact Lie group.
H. Hopf determined in 1941 that the real homology of simple compact Lie groups is that of a product of odd spheres ...
The exponents of a Lie group are the numbers i such
that S(2i+1) is an allowed sphere ...
neither the U-series nor the Sp-series have torsion.
The exponents ... for U(n) ... are 0, 1, ... , n-1 ... and jump by two in Sp(n).

But for the orthogonal series one has to consider some Stiefel manifolds instead of spheres, which have the same real homology ...
It ... introduces (preciesely) 2-torsion:
in fact, Spin(n), n≥7 and SO(n), n≥3, have 2-torsion.
The low cases Spin(3,4,5,6) coincide
with Sp(1), Sp(1)xSp(1), Sp(2) and SU(4) , and have no torsion.

For ... G2 ... SU(2) -> G2 -> M11 ... where M11 is again a Steifel manifold, with real homology like S11, but with 2-torsion ...

For F4 we do not get the sphere structure from any irrep,
and in fact F4 has 2- and 3-torsion. ...

2- and 3-torsion appears in ... E6 and E7 ...

E8 has 2-, 3- and 5-torsion ...
The Coxeter number of (dim - rank) of E8 is 30 = 2 x 3 x 5 ,
in fact a mnemonic for the exponents of E8 is:
they are the coprimes up to 30, namely (1,7,11,13,17,19,23,29) ...
The first perfect numbers are 6, 28, and 492,
associated to the primes 2, 3 and 5 (... Mersenne numbers ...) ...
496 = dim O(32) = dim E(8) x E(8) . Why the square?

It also happens in O(4) , dim = 6 (prime 2), as O(4) ...[is like]... O(3) x O(3) ;
even O(8) [dim = 28] (prime 3) is like S7 x S7 x G2 ...

The sphere structure of compact simple Lie groups has a curious "capicua" ...
Catalan word (cap i cua 0 = head and tail) ... form:
the exponents are symmetric from each end; for example ...

exponents of E6: 1,4,5,7,8,11. Differences: 3,1,2,1,3

exponents of E7: 1,5,7,9,11,13,17. Differences: 4,2,2,2,2,4 ...

exponents of E8 ... 1,7,11,13,17,19,23,29 ...[Differences 6,4,2,4,2,4,6]...

The real homology algebra of a simple Lie group is a Grassmann algebra,
as it is generated by odd (i.e., anticommutative) elements.
However, from them we can get, in the enveloping algebra,
multilinear symmetric forms, one for each generator; ...
in physics they are called Casimir invariants,
in mathematics the invariants of the Weyl group ...".

Martin Cederwall and Jakob Palmkvist, in "The octic E8 invariant"
hep-th/0702024, say:

"... The largest of the finite-dimensional exceptional Lie groups, E8, with Lie algebra e8, is an interesting object ... its root lattice is the unique even self-dual lattice in eight dimensions (in euclidean space, even self-dual lattices only exist in dimension 8n). ... Because of self-duality, there is only one conjugacy class of representations, the weight lattice equals the root lattice, and there is no "fundamental" representation smaller than the adjoint. ...
Anything resembling a tensor formalism is completely lacking. A basic ingredient in a tensor calculus is a set of invariant tensors, or "Clebsch-Gordan coefficients". The only invariant tensors that are known explicitly for E8 are the Killing metric and the structure constants ...

The goal of this paper is to take a first step towards a tensor formalism for E8 by explicitly constructing an invariant tensor with eight symmetric adjoint indices. ...

On the mathematical side, the disturbing absence of a concrete expression for this tensor is unique among the finite-dimensional Lie groups. Even for the smaller exceptional algebras g2, f4, e6 and e7, all invariant tensors are accessible in explicit forms, due to the existence of "fundamental" representations smaller than the adjoint and to the connections with octonions and Jordan algebras. ...

The orders of Casimir invariants are known for all finite-dimensional semi-simple Lie algebras. They are polynomials in U(g), the universal enveloping algebra of g, of the form t_(A1...Ak) T^(A1 . . . TAk), where t is a symmetric invariant tensor and T are generators of the algebra, and they generate the center U(g)^(g) of U(g).

The Harish-Chandra homomorphism is the restriction of an element in U(g)^(g) to a polynomial in the Cartan subalgebra h, which will be invariant under the Weyl group W(g) of g.

Due to the fact that the Harish-Chandra homomorphism is an isomorphism
from U(g)^(g) to U(h) W(g) one may equivalently consider
finding a basis of generators for the latter, a much easier problem.
The orders of the invariants follow more or less directly
from a diagonalisation of the Coxeter element,
the product of the simple Weyl reflections ...

In the case of e8, the center U(e8)^(e8) of the universal enveloping subalgebra is generated by elements of orders 2, 8, 12, 14, 18, 20, 24 and 30.
The quadratic and octic invariants correspond to primitive invariant tensors in terms of which the higher ones should be expressible. ...
the explicit form of the octic invariant is previously not known ...

E8 has a number of maximal subgroups, but one of them, Spin(16)/Z2, is natural for several reasons.
Considering calculational complexity,
this is the subgroup that leads to the smallest number of terms in the Ansatz.

Considering the connection to the Harish-Chandra homomorphism,
K = Spin(16)/Z2 is the maximal compact subgroup of the split form G = E8(8).

The Weyl group is a discrete subgroup of K,
and the Cartan subalgebra h lies entirely in the coset directions g/k ...

We thus consider the decomposition of the adjoint representation of E8 into representations of the maximal subgroup Spin(16)/Z2.

The adjoint decomposes into the adjoint 120 and a chiral spinor 128. ...

Our convention for chirality is GAMMA_(a1...a16) PHI = + e_(a1...a16) PHI .

The e8 algebra becomes (2.1)

[T^(ab) , T^(cd)] = 2 delta^([a)_([c) T^(b])_(d]) ,
[T^(ab) , PHI^(alpha)] = (1/4) (GAMMA^(ab) PHI)^(alpha) ,
[PHI^(alpha) , PHI^(alpha)] = (1/8) (GAMMA_(ab))^(alpha beta) T^(ab) ,

... The coefficients in the first and second commutators are related
by the so(16) algebra. The normalisation of the last commutator is free,
but is fixed by the choice for the quadratic invariant, which for the case above is

X2 = (1/2) T_(ab) T^(ab) + PHI_(alpha) PHI^(alpha) .

Spinor and vector indices are raised and lowered with delta .
Equation (2.1) describes the compact real form, E8(-248) .

By letting PHI -> i PHI one gets E8(8),
where the spinor generators are non-compact,
which is the real form relevant as duality symmetry in three dimensions
(other real forms contain a non-compact Spin(16)/Z2 subgroup).

The Jacobi identities are satisfied thanks to the Fierz identity

(GAMMA_(ab)_[(alpha beta) (GAMMA_(ab)_(alpha beta)] = 0 ,

which is satisfied for so(8) with chiral spinors, so(9), and so(16) with chiral spinors
(in the former cases the algebras are so(9), due to triality, and f4).

The Harish-Chandra homomorphism tells us that the "heart" of the invariant
lies in an octic Weyl-invariant of the Cartan subalgebra.
A first step may be to lift it to a unique Spin(16)/Z2-invariant in the spinor,
corresponding to applying the isomorphism fÅ|1 above.
It is gratifying to verify ... that there is indeed an octic invariant
(other than (PHI PHI)^4), and that no such invariant exists at lower order. ...

Forming an element of an irreducible representation containing a number of
spinors involves symmetrisations and subtraction of traces, which can be rather
complicated. This becomes even more pronounced when we are dealing with
transformation ... under the spinor generators, which will transform as spinors.

Then irreducibility also involves gamma-trace conditions. ... The transformation ... under the action of the spinorial generator is an so(16) spinor. The vanishing of this spinor is equivalent to e8 invariance. The spinorial generator acts similarly to a supersymmetry generator on a superfield ... The final result for the octic invariant is, up to an overall multiplicative constant:

$$
\begin{aligned}
X_8 = {} & \tfrac{1}{3072}\varepsilon^{a_1\cdots a_{16}}T_{a_1a_2}\dots T_{a_{15}a_{16}} \\
& \quad - 30\mathrm{tr}T^8 + 14\mathrm{tr}T^6\mathrm{tr}T^2 + \tfrac{35}{4}(\mathrm{tr}T^4)^2 - \tfrac{35}{8}\mathrm{tr}T^4(\mathrm{tr}T^2)^2 + \tfrac{15}{64}(\mathrm{tr}T^2)^4 \\
& + \big[2\mathrm{tr}T^6 - \mathrm{tr}T^4\mathrm{tr}T^2 + \tfrac{1}{8}(\mathrm{tr}T^2)^3\big](\phi\phi) \\
& \quad + \big[(\tfrac{5}{4}\mathrm{tr}T^4 - \tfrac{1}{2}(\mathrm{tr}T^2)^2)T^{ab}T^{cd} + \tfrac{27}{4}\mathrm{tr}T^2T^{ab}(T^3)^{cd} \\
& \qquad - 15T^{ab}(T^5)^{cd} - 15(T^3)^{ab}(T^3)^{cd}\big](\phi\Gamma_{abcd}\phi) \\
& \quad + \big[\tfrac{1}{16}\mathrm{tr}T^2T^{ab}T^{cd}T^{ef}T^{gh} - \tfrac{5}{8}T^{ab}T^{cd}T^{ef}(T^3)^{gh}\big](\phi\Gamma_{abcdefgh}\phi) \\
& \quad - \tfrac{1}{192}T^{ab}T^{cd}T^{ef}T^{gh}T^{ij}T^{kl}(\phi\Gamma_{abcdefghijkl}\phi) \\
& + \big[7\mathrm{tr}T^4 - \tfrac{31}{8}(\mathrm{tr}T^2)^2\big](\phi\phi)^2 \\
& \quad - \tfrac{3}{64}T^{ab}T^{cd}T^{ef}T^{gh}(\phi\phi)(\phi\Gamma_{abcdefgh}\phi) \\
& \quad + \big[\tfrac{5}{64}T^{ab}T^{cd}T^{ef}T^{gh} - \tfrac{15}{16}T^{ab}T^{ce}T^{df}T^{gh} \\
& \qquad + \tfrac{5}{8}T^{ae}T^{bf}T^{cg}T^{dh}\big](\phi\Gamma_{abcd}\phi)(\phi\Gamma_{efgh}\phi) \\
& \quad + \big[\tfrac{3}{2}(T^3)^{ab}T^{cd} - \tfrac{1}{8}\mathrm{tr}T^2T^{ab}T^{cd}\big](\phi\phi)(\phi\Gamma_{abcd}\phi) \\
& \quad + \big[\tfrac{15}{16}(T^3)^{ab}T^{cd} - \tfrac{3}{16}\mathrm{tr}T^2T^{ab}T^{cd} + \tfrac{5}{4}(T^2)^{ac}(T^2)^{bd}\big](\phi\Gamma_{ab}{}^{ij}\phi)(\phi\Gamma_{cdij}\phi) \\
& \quad + \tfrac{15}{8}T^{ab}T^{cd}(T^2)^{ef}(\phi\Gamma_{abe}{}^{i}\phi)(\phi\Gamma_{cdfi}\phi) \\
& + \tfrac{1}{2}\mathrm{tr}T^2(\phi\phi)^3 + \tfrac{55}{32}T^{ab}T^{cd}(\phi\phi)^2(\phi\Gamma_{abcd}\phi) \\
& \quad + \tfrac{1}{8}T^{ab}T^{cd}(\phi\phi)(\phi\Gamma_{ab}{}^{ij}\phi)(\phi\Gamma_{cdij}\phi) \\
& \quad + \big[-\tfrac{1}{384}T^{ab}T^{cd} + \tfrac{7}{192}T^{ac}T^{bd}\big](\phi\Gamma_{ab}{}^{ij}\phi)(\phi\Gamma_{cd}{}^{kl}\phi)(\phi\Gamma_{ijkl}\phi) \\
& - \tfrac{57}{32}(\phi\phi)^4 + \tfrac{1}{12288}(\phi\Gamma_{ab}{}^{cd}\phi)(\phi\Gamma_{cd}{}^{ef}\phi)(\phi\Gamma_{ef}{}^{gh}\phi)(\phi\Gamma_{gh}{}^{ab}\phi) \\
& + \beta[-\tfrac{1}{2}\mathrm{tr}T^2 + (\phi\phi)]^4\ .
\end{aligned}
\tag{2.3}
$$

Here, β is an arbitrary constant multiplying the fourth power of the quadratic invariant. The trace vanishes for $\beta = \frac{9}{127}$ (that such a value exists at all is non-trivial and provides a further check on the coefficients). The occurrence of the prime 127 is not incidental; taking the trace of $\delta^{(AB}\delta^{CD}\delta^{EF}\delta^{GH)}$ gives $\delta_{GH}\delta^{(AB}\delta^{CD}\delta^{EF}\delta^{GH)} = (\frac{1}{7}\cdot 248 + \frac{6}{7})\delta^{(AB}\delta^{CD}\delta^{EF)} = \frac{2\cdot 127}{7}\delta^{(AB}\delta^{CD}\delta^{EF)}$. The actual technique we use for calculating the trace is not to extract the eight-index tensor, but to act on the invariant with $\frac{1}{2}\frac{\partial}{\partial T_{ab}}\frac{\partial}{\partial T^{ab}} + \frac{\partial}{\partial\phi_a}\frac{\partial}{\partial\phi^a}$. We remind that eq. (2.3) gives the octic invariant for the compact form $E_{8(-248)}$. The corresponding expression for the split form $E_{8(8)}$ is obtained by a sign change of the terms containing ϕ^{4k+2}.

...”.